PRINCIPLES

OF

REINFORCED CONCRETE CONSTRUCTION

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THIRD EDITION, REVISED AND ENLARGED

TOTAL ISSUE, TWENTY THOUSAND

NEW YORK

JOHN WILEY & SONS, Inc.

London: CHAPMAN & HALL, Limited

1919

PRESS OF
BRAUNWORTH & CO.
BOOKBINDERS AND PRINTERS
BROOKLYN, N. Y.

PREFACE TO THE THIRD EDITION.

In the preparation of this edition, the general purposes set forth in the preface to the first edition have been adhered to, namely, "to present in a systematic manner those principles in mechanics underlying the design of reinforced concrete; to present the results of all available tests that will aid in establishing coefficients in working stresses; and to give such illustrative material from actual designs as will serve to make clear the principles involved."

As compared to the second edition, the material has been considerably amplified in several respects and rearranged with a view to more convenient use as a textbook. Separate chapters are devoted to the theory of flexure; bond and shear; and the design of beams; and the practice has been followed in these, as well as other fundamental chapters, of including both the theoretical and the experimental treatment in the same chapter. All available experimental data have been carefully reviewed and such results presented as are most reliable and significant in connection with the subjects in question.

A new chapter has been added covering the analysis of flat slabs, including slabs supported on beams, as well as the so-called flat slab system of floor construction. The chapter on building construction has been extended to include a more detailed treatment of continuous beams and girders. The tables and diagrams are collected in a chapter at the end of the work. The chapters illustrating the applications of reinforced concrete have been very slightly changed. To introduce illustrative material which

would in any sense be complete would require a very large amount of space and would go far beyond the purpose of the work. For such illustrative material the reader is referred to the current engineering periodicals.

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MADISON, WIS.,
Sept. 1, 1919.

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REINFORCED-CONCRETE CONSTRUCTION.

CHAPTER I.

INTRODUCTORY.

1. **Historical Sketch.**—The invention of reinforced concrete is usually credited to Joseph Monier, but his first constructions are antedated by those of Lambot, who in 1850 constructed a small boat of reinforced concrete and in 1855 exhibited the same at the Paris Exposition. In this latter year Lambot took out patents on this form of construction; it was regarded by him as especially well adapted to shipbuilding, reservoir work, etc.

In 1861, Monier, who was a Parisian gardener, constructed tubs and tanks of concrete surrounding a framework or skeleton of wire. In the same year Coignet announced his principles for reinforcing concrete, and proposed construction of beams, arches, pipes, etc. Both he and Monier executed some work in the new material at the Paris Exposition of 1867. In this year Monier took out patents on his reinforcement. It consists of two sets of parallel bars, one set at right angles to and lying upon the other, thus forming a mesh of bars. This system, and slight modifications of it, are extensively used at the present time, particularly for slab reinforcement. Though even the early Monier patents covered principles of wide application, still the early work in reinforced concrete was confined to a comparatively narrow field.

In 1884–5 the German and American rights of the Monier patents fell into the hands of German engineers. One of these, G. A. Wayss, and J. Bauschinger at once began an experimental investigation of the Monier system, and in 1887 they published their findings. The investigation proved reinforced concrete a valuable means of construction, and furnished some formulas and methods for design. From this time on, the use of reinforced concrete in Austria spread rapidly, and a few years ago the engineers of that country were credited with having done more to develop the new construction than those of any other country. Among these engineers should be mentioned Melan, who in the early 90's originated a system in which I or T beams are the principal element of strength, providing compressive as well as tensile strength. In Germany government regulations hindered the application of reinforced concrete for a time, but now it is widely used in that country. Over two hundred systems of reinforcement, it has been stated, have been developed in Germany alone.

In France the Monier system was never developed as in countries already mentioned. Here, as elsewhere, many other systems of reinforcement were invented from time to time, among which should be mentioned that of Hennebique, who was probably the first to use stirrups and "bent-up" bars. This system is in general use, and the elements of Hennebique's system are probably more widely used than those of any other.

In England and America the first use of iron or steel with concrete arose in the effort to fireproof the former by means of the latter. Attempting to utilize also the strength of concrete, Hyatt built beams of concrete reinforced with metal in various ways, and with Kirkaldy of London performed tests on such beams and published the results of the investigation in 1877. The first reinforced-concrete work in the United States was done in 1875 by W. E. Ward, who constructed a building in New York state in which walls, floor-beams, and roof were made of concrete reinforced with metal to provide tensile strength. But the Pacific Coast saw the actual early development of this form

of construction. H. P. Jackson, G. W. Percy, and E. L. Ransome were the pioneer workers. Jackson has been credited with reinforced constructions dating as far back as 1877, but Ransome executed the most notable early examples. Among these are a warehouse (1884 or '85), a factory building a few years later, the building of the California Academy of Science (1888 or '89), and the museum building of Leland Stanford Junior University (1892). Percy was the architect of the last two. The museum building contains spans of 45 feet and is reinforced throughout. This and the Academy building withstood the recent earthquake remarkably well—the museum better than its two brick annexes.

Other pioneer constructors in reinforced concrete in this country were F. von Emperger and Edwin Thacher. The former introduced the Melan system (1894) and built the first reinforced arch bridges of considerable span. Thacher also was—and still is—a bridge-builder. His first large reinforced-concrete bridge was built in 1896 and was without precedent here or in Europe.

America is the home of the "patent bar". Both Ransome and Thacher invented bars known by their respective names, the patented feature of which is to furnish a "grip" between bar and concrete; besides these two there are several others on the market designed to give additional grip or bond. There are also patented bars for supplying "shear reinforcement". Some of these forms have been introduced into Europe.

Reinforced-concrete construction has had a remarkable development in the past twenty years and is now regarded by engineers and architects generally as a form of construction to be used whenever considerations of economy demand it. Uncertainties of behavior and of theory which formerly existed to a considerable extent have been largely eliminated and design in reinforced concrete is now carried out on rational principles in the same manner as in other materials, although in the nature of the case the variation in quality is much greater than in such material as structural steel.

2. Use and Advantages of Reinforced Concrete.—A combination of steel and concrete constitutes a form of construction possessing to a large degree the advantages of both materials without their disadvantages. It will be desirable at the outset to consider briefly these advantages in order better to appreciate the field in which this type of construction is likely to be most successful.

Steel is a material especially well suited to resist tensile stresses, and for such purposes the most economical form—the solid compact bar—is well adapted. To resist compressive stresses steel must be made into more expensive forms, consisting of relatively thin parts widely spread, in order to provide the necessary lateral rigidity. A serious disadvantage in the use of steel in many locations is its lack of durability; and, again, a comparatively low degree of heat destroys its strength, thus rendering it necessary to add a protective covering where a fire-proof structure is demanded. Steel is a relatively expensive building material, and its cost tends to increase.

Concrete is characterized by low tensile strength, relatively high compressive strength, and great durability. It is a good fire-proof material, and therefore serves as a good fire-proof covering for steel. It is also found that steel well covered by concrete is thoroughly protected from corrosion. Concrete is also a comparatively cheap material and is readily available in almost any locality.

In the design of structural members these qualities of steel and concrete will lead to the use of the two materials about as follows: For those structural members carrying purely tensile stresses steel must be employed, but it may be surrounded by concrete as a protection against corrosion and fire, or merely for the sake of appearance. For those members sustaining purely compressive stresses concrete is fundamentally the better and cheaper material. With concrete costing 30 cents per cubic foot, for example, and steel 4 cents per pound, or about \$20.00 per cubic foot, and with working stresses of 400 and 15,000 lbs/in^2, respectively, the relative cost of the

two materials for carrying a given load is as $\frac{30}{400}$ is to $\frac{2000}{15,000}$, or as 45 is to 80. For large and compact compressive members plain concrete will therefore naturally be used, especially where durability is a factor. For more slender members, however, such as long columns, plain concrete is too brittle a material, and therefore too much affected by secondary and unknown stresses to be satisfactory; and for such members steel alone, or the two materials in combination, will preferably be used. Steel may be used with concrete in the form of small rods to reinforce the concrete; or it may be used in larger sections and simply surrounded and held rigidly in place by the concrete, most of the load being carried by the steel; or, finally, a steel column may be used and merely fireproofed by the concrete. As the cost of steel in the form of rods is much less than in the form of built members, and as compressive stresses can, in general, be carried more cheaply by concrete than by steel, economical construction will lead to the use of the maximum amount of concrete and the minimum amount of steel consistent with safety, although this principle will be modified by various practical considerations.

For these structural forms in which both tension and compression exist, that is to say, in all forms of beams, the combination of the two materials is particularly advantageous. Here the tensile stresses are carried by steel rods embedded in the concrete near the tension side of the beam. The steel is thus used in its cheapest form, it is thoroughly protected by the concrete, and the compressive stresses are carried by the concrete. Concrete alone cannot be used to any appreciable extent to carry bending stresses on account of its low and uncertain tenacity, but a concrete beam with steel rods embedded in it to carry the tensile stresses is a strong, economical, and very durable form of structure.

From these considerations it follows that reinforced-concrete construction is advantageous to varying degrees in different types of structures. Some of the most important of

these types will here be noted, together with the advantages accompanying the use of reinforced concrete in their design.

3. *Buildings.*—This type of construction is especially useful for floor-slabs and to a somewhat less degree for beams, girders, and columns. It is also well adapted for footings in foundations, being more economical than I-beam footings embedded in concrete.

4. *Culverts and small Girder Bridges.*—Very satisfactory on account of its simplicity and economy as compared to masonry arches, and because of its durability as compared to steel bridges.

5. *Retaining-walls, Dams, and Abutments.*—Often economical for such structures as compared to ordinary masonry. Plain masonry structures of this kind are designed to resist lateral forces by their weight alone, the resulting compressive stresses, except in extremely large structures, being very small and much below safe values. By the use of reinforced concrete these structures can be designed of a more economical type and so arranged as to utilize the concrete in the form of beams, thus developing more nearly the full compressive strength of the material. The steel reinforcement is fully protected from corrosion, a factor which prevents the use of all-steel frames for structures of this class.

6. *Arch Bridges.*—In this form of structure reinforced concrete possesses less advantage over ordinary masonry than in those forms where the compressive stresses are less important. In an arch the stresses are principally compressive, and these do not require steel reinforcement; it is only to provide for the relatively small bending stresses due to moving loads, or as a precaution against undesirable cracks, that steel is serviceable. No large economy can be obtained through its use. By reason of greater simplicity and the less expensive abutments required, a flat-top culvert or beam bridge, with abutments of reinforced concrete, is more advantageous for short spans than the arch.

7. *Reservoir Walls, Floors, and Roofs.*—Very well adapted as a durable material and lending itself to lighter design than common masonry.

8. *Conduits and Pipe Lines.*—Reinforced concrete can often be used to great advantage in a water-conduit or large sewer. It is also sometimes used for pipe lines and tanks under pressure, the steel being relied upon to resist the tensile stresses, while the concrete serves as a protection and as a water-tight covering. The amount of steel may thus be determined by considerations of strength alone, where otherwise a much larger amount of metal would be needed and in a more expensive form.

9. *Elevated Tanks, Bins, etc.*—Advantageous because of its durability and its adaptability in the construction of heavy floors and walls subjected to lateral pressure. Of especial value for coal-bins, either for flooring and lining alone, or for the entire structure.

10. *Chimneys and Towers.*—Possesses advantages over brick or stone masonry in the fact that it forms a structure of monolithic character, resulting in greater certainty in the stresses and economy in design.

11. *Piles, Railroad Ties, etc.*—The use of a moderate amount of steel with concrete so as to give to this material a reliable tensile and bending resistance has opened the way for its use in a great variety of forms, not only as complete structures, or important members of structures, but also in many special individual forms. Concrete piles are valuable substitutes for piles of wood where the latter would be subject to deterioration. Reinforced concrete has obvious advantages as a material for railroad ties, but a successful design has not yet been developed. This material is also well adapted to many other special uses, such as fence posts, transmission line poles and all similar purposes where the structure is exposed to the action of the elements.

CHAPTER II.

PROPERTIES OF THE MATERIALS.

12. In a design where two or more materials are combined in the same member the stresses in the different materials depend upon the elastic properties of the materials as well as upon the superimposed loads. Therefore in making such designs a knowledge of these elastic properties is quite as necessary as a knowledge of the strength of the materials.

CONCRETE.

13. General Requirements.—The conditions to be met in reinforced-concrete construction require the use, generally, of a concrete of relatively high grade. In this type of construction the strength of the material is of much greater importance than it is in many forms of plain concrete design, as the dimensions of the structures are more directly dependent upon strength and less upon weight. A comparatively strong concrete is therefore found to be economical.

It is especially important, also, that the concrete be of uniform quality and free from voids, as the sections are comparatively small and the stability of the structure, to a much greater extent than is the case with massive concrete, is dependent upon the integrity of every part. Thoroughly sound concrete is also required in order to insure good adhesion to the steel reinforcement and adequate protection of the steel from corrosion and from fire. These requirements call for great care in the preparation and placing of the material.

Concrete is subject to great variations in its properties, owing to the great variations in the character and proportions of its ingredients and in its preparation. It is therefore difficult to judge from results of tests made under certain conditions as to what may fairly be expected of a concrete prepared under other conditions. For this reason it is important that special tests be made with the materials actually to be used on the work. Regular and systematic tests should also be made during the progress of construction to serve as a check on the preliminary tests and to prevent any deterioration of quality due to possible changes in materials or in the method of mixing and placing.

14. Cement.—Portland cement only should be used; it should meet such standard specifications as those of the American Society for Testing Materials. The rapidity of hardening of different cements varies considerably and may be an element requiring special attention where the structure is to receive its load very early or where such load is to be long deferred.

15. Fine Aggregate.—The sand, or fine aggregate, should be clean and preferably of coarse grain. It should, if possible, meet the specifications for aggregates of the American Society for Testing Materials. A fine sand requires more cement than a coarse sand for equal strength, and more water for a like consistency. In the case of a very fine sand the difference may be very marked, so that unless care is taken and special tests made, the resulting concrete is likely to be porous and deficient in strength and adhesive power. Where the use of fine sand is contemplated, tests of strength may show that a considerable extra cost may be justified in securing a coarser material. The effect of size of sand is shown in Art. 20.

16. Coarse Aggregate.—For the coarse aggregate (material exceeding $\frac{1}{4}$ inch in size) either broken stone or gravel is satisfactory. In either case the material should be screened to remove the dust or sand, and to remove particles larger than the maximum size desired. Beyond this, the screening of stone to

size is undesirable unless an artificial mixture is to be made, as it tends to increase the proportion of voids. Pit-run gravel is rarely sufficiently uniform and correct in proportions to be safely used without screening and remixing.

The maximum desirable size of stone or gravel depends upon the size of the structural forms and the size and spacing of the reinforcement, it being desirable to use as large a size of aggregate as will admit of convenient working. Maximum sizes of stone of $\frac{3}{4}$ inch to $1\frac{1}{4}$ inches are common, but on heavy work, with rods widely spaced, there is no objection to still larger sizes. Generally speaking, for sizes below 1 inch the density and strength of the concrete increases somewhat with size of coarse aggregate.

The crushing strength of a gravel concrete is usually a little less than one of broken stone of the same proportion of voids, but the difference is not great. On the other hand gravel concrete flows more readily and is somewhat easier to mix and place, so that a greater density is likely to be secured. The difference in tensile strength is not well determined, but the few tests available indicate about the same relative difference as in compressive strength.

17. Proportions of Ingredients.—The proportions commonly used vary from about $1:1\frac{1}{2}:3$ to $1:3:6$ of cement, fine and coarse aggregate respectively. Richer mixtures than $1:2:4$ are not common, nor poorer mixtures than $1:2\frac{1}{2}:5$, although with well-graded material a very satisfactory concrete can be made of $1:3:6$ proportions. Occasionally where the design is determined by other considerations than strength and cost, (as in the case of columns in a building), a very rich mixture or a poor one may be desirable, but where these elements determine the design, the most economical concrete will be a rich concrete of about the proportions above indicated. Customary proportions, such as $1:2:4$, should not be blindly adopted. In any important work a careful study of the materials and of the best proportions to use for economy and strength will be well repaid. To secure sound and reliable work, with good

adhesion and tensile strength, there must be no unfilled voids in the stone and little or none in the sand. The former is of more importance than the latter, and if cost and strength are to be reduced it should be done by using a poorer mortar to fill the voids in the stone. For equal amounts of cement, the denser the mixture (or the smaller the percentage of voids) the stronger the concrete.

18. Consistency.—The general practice at the present time is to use a much wetter consistency than formerly. To a certain extent this is necessary and desirable, but the ease of manipulation secured by a liberal use of water has led to the employment, in many cases, of such wet mixtures that the integrity and strength of the concrete is far below what it might and should be. A relatively dry concrete, thoroughly rammed, will have a maximum strength, but such a material cannot be used in practice for reinforced work. A concrete of semi-plastic or quaking consistency will, in time, acquire about the same strength as the drier mix, but while it is possible to place such concrete around reinforcing bars by a considerable use of tampers, it requires a large amount of hand work, and in practice a still wetter mix must be used. The best practicable consistency for most work is best described by the word "mushy." Such a concrete will flow into place in the larger sections of the work, requiring only spading to secure a smooth surface next to the forms; but in the smaller sections, such as beams, it will require some shoving and spading to bring it into place around the reinforcement. The use of enough water to give a "sloppy" consistency, enabling the material to flow like water, is very objectionable. The strength is very considerably impaired and segregation of the coarse aggregate in the carts or in the forms is likely to occur.

Concrete of "mushy" consistency will be somewhat weaker than the ideal compacted concrete, but under actual working conditions, will be much more reliable and will be free from voids. In the case of reinforced work reliability is more important than maximum strength, and is promoted by using

concrete of such consistency that it can readily be worked into place in the forms and around the reinforcing steel.

19. Mixing of Concrete.—It is essential that the mixing be thoroughly done. Machine mixing should be required wherever practicable and the time of mixing or number of turns and rate specified. With the ordinary types of rotary mixers at least 1½ minutes should be required, and still better results can be obtained with longer periods. A mixing of 10 to 20 seconds, as is not uncommon, is wholly inadequate.

20. Compressive Strength.—The compressive strength of concrete is dependent upon many factors so that it is difficult, and at the same time somewhat misleading to present "average values." Obviously, in any important work, the strength should be determined, if possible, under the actual conditions under which the concrete is to be used. Uniformity is quite as important as high average strength.

The strength of concrete depends primarily upon: (*a*) the amount of cement used, (*b*) the density of the concrete, (*c*) the proportion of water used, and (*d*) the age of the specimen. Within the usual proportions the strength will be about proportional to the amount of cement employed. Density is greatly affected by the size and gradation of the particles of the aggregate, and for like proportions of cement the strength increases markedly with the density.

The shape of the specimen has also a considerable effect upon the test results. Formerly the cubical specimen was the standard form, but results of the extensive study made on concrete in recent years have led to the adoption of the cylindrical form, of about two diameters height, as the standard. The standard age for testing is usually thirty or sixty days, the latter being preferable and giving about 75% of the strength at six months. Beyond six months the concrete will continue to increase somewhat in strength, but at a relatively slow rate.

One of the best series of tests is that made at the Watertown Arsenal for Mr. George A. Kimball, Chief Engineer of

the Boston Elevated Railway Company.* The concrete was made of five brands of Portland cement, coarse, sharp sand, and broken stone up to $2\frac{1}{2}$ inches size. The concrete was well rammed into the molds, water barely flushing to the surface. The specimens were buried in wet ground after being taken from the molds. The average results were as follows:

TABLE NO. 1.

COMPRESSIVE STRENGTH OF CONCRETE CUBES.

WATERTOWN ARSENAL, 1899.

Mixture.	Brand of Cement.	Strength, Pounds per Square Inch.			
		7 Days.	1 Month.	3 Months.	6 Months.
1:2:4	Saylor	1724	2238	2702	3510
	Atlas	1387	2428	2966	3953
	Alpha	904	2420	3123	4411
	Germania	2219	2642	3082	3643
	Alsen	1592	2269	2608	3612
	Average	1565	2399	2896	3826
1:3:6	Saylor	1625	2568	2882	3567
	Atlas	1050	1816	2538	3170
	Alpha	892	2150	2355	2750
	Germania	1550	2174	2486	2930
	Alsen	1438	2114	2349	3026
	Average	1311	2164	2522	3088

The above table shows fairly well the increase of strength with age. On the average the strength at 7 days is about 40%, 1 month 65 to 70%, and 3 months 75 to 80% of the strength at six months.

Table No. 2 gives average values of compressive strength of 1 : 2 : 4 concrete from tests on 8×8-inch cubes and 8×16-inch cylinders, as determined by three different laboratories. The consistency was soft but not sloppy.†

* Tests of Metals, 1899, p. 717.

† Jour. Am. Conc. Inst., Vol. II, 1914, p. 430.

TABLE No. 2.

COMPRESSIVE STRENGTH OF CONCRETE CUBES AND CYLINDERS.

Concrete 1 : 2 : 4; age 3 months.

Laboratory.	Univ. of Ill.	Univ. of Wis.	Mass. Inst. Tech.	Average.
	lbs/in²	lbs/in²	lbs/in²	lbs/in²
8×8-inch cubes....	3521	3135	3917	3524
8×16-inch cyl......	2769	2370	2934	2691
Ratio, cyl : cube...	.79	.76	.75	.764

These values illustrate what may readily be obtained with good material.

Results of a large number of tests made by the Bureau of Standards gave a ratio of about .80 for the strength of cylinders compared to cubes.*

The effect of an excessive amount of water is well shown in Fig. 1, representing results obtained at the University of Illinois on 6-inch cylinders of 1 : 2 : 4 concrete 3 months old.† The "normal consistency" was somewhat soft, it being such that a cylinder of fresh concrete would just begin to slump upon removal of forms. Other tests showed the normal consistency to give as strong result in two months as a dry consistency This diagram also shows the importance of providing sufficient moisture during curing.

TABLE No. 3.

EFFECT OF CONSISTENCY ON COMPRESSIVE STRENGTH OF 1 : 2 : 4 LIMESTONE CONCRETE.

(University of Wisconsin.)

Consistency.	Per Cent Water by Weight.	Compressive Strength in Pounds per Sq. In.		
		14 Days.	70 Days.	350 Days.
Dry..............	6	1774	2635	4000
Quaking...........	7	1945	3126	4320
"Mushy".........	8	1709	2927	4500
"Soupy".........	10	1283	2578	3070

* Tech. Papers No. 2, 1912.

† Jour. Am. Concr. Inst., Vol. II, 1914, p. 437.

Table No. 3 gives additional data on the effect of consistency. This table, taken together with Fig. 1, shows that the strength of a "mushy" consistency is about the same as a drier one,

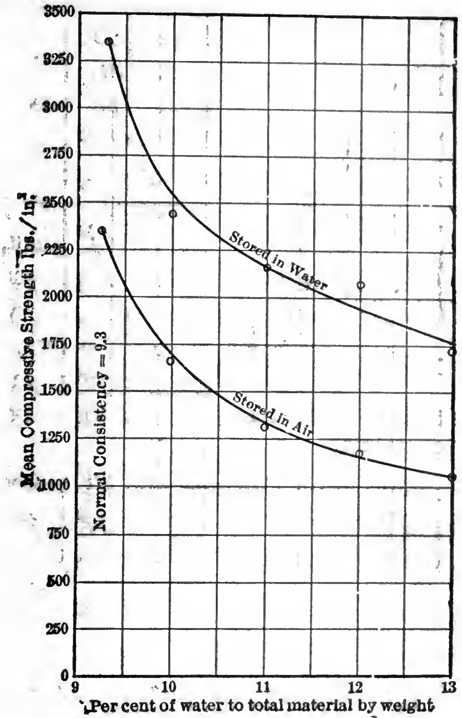


FIG. 1.—Effect of Consistency on Compressive Strength.

but that an increase of water beyond this point results in rapidly diminishing strength.

The size and gradation of the sand or fine aggregate has a very great influence on density and strength. The effect of size of sand has been thoroughly investigated by Feret. Fig. 2, from Johnson's "Materials of Construction," shows results obtained by Feret on 1 : 3 mortar cubes after hardening one year in fresh water. The sand used consisted of mixtures of various proportions of fine (.0 to .5 mm.), medium (.5 to 2 mm.), and coarse (2 to 5 mm.), sand, and in the figure the result from

any particular mortar is recorded in the triangle at such distances from the three base-lines as will represent the proportions of each size sand used. Lines of equal strength were then drawn in the diagram. Thus the strength of the mortar in which only fine sand was used was only 1400 lbs/in². The maximum strength of 3500 lbs/in² was obtained from a mixture containing about 85% of coarse sand and 15% of fine, with a very little sand of medium size. This diagram shows in a striking manner the effect of size of sand.

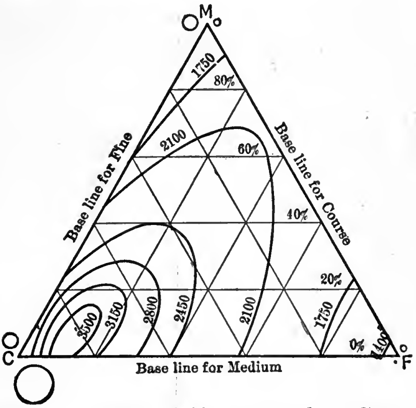


FIG. 2.—Effect of Size of Sand on Compressive Strength.

The Joint Committee on Concrete and Reinforced Concrete recommends the following values for strength of cylindrical specimens at 28 days as the maximum to be used in design:

TABLE NO. 4.

MAXIMUM COMPRESSIVE STRENGTH OF CONCRETE AS RECOMMENDED BY THE JOINT COMMITTEE.

Based on cylindrical specimens 8×16 inches; age 28 days. Pounds per square inch.

Aggregate.	Ratio Cement to Sum of Fine and Coarse Aggregates Measured Separately.				
	1 : 3	1 : 4½	1 : 6	1 : 7½	1 : 9
Granite, trap rock.........	3300	2800	2200	1800	1400
Gravel, hard limestone, and hard sandstone..........	3000	2500	2000	1600	1300
Soft limestone and sandstone	2200	1800	1500	1200	1000
Cinders..................	800	700	600	500	400

21. Tensile Strength.—The tensile strength of concrete is quite as important as the compressive strength. In fact the

most common type of failure of a reinforced concrete beam is closely related to the tensile strength of the concrete. The tensile strength is generally from one-eighth to one-twelfth of the compressive strength, but this ratio varies considerably. The character of the material and workmanship has probably a greater influence upon the tensile strength than upon the compressive.

Tests made by Mr. M. O. Withey on 1 : 2 : 4 concrete, 28 days old, gave results averaging 189 lbs/in², varying from 142 to 160 lbs/in². The compressive strength was 1940 lbs/in².* Tests by Mr. W. H. Henby † gave results as follows:

Mixture.	Compressive Strength.	Tensile Strength.
1 : 2 : 4	3000 lbs/in²	180 lbs/in²
1 : 3 : 6	1800 "	115 "

Tests by Professor W. K. Hatt ‡ gave the following results:

Kind of Concrete.	Age, days.	Compressive Strength, lbs/in².	Tensile Strength, lbs/in².
1:2:4 (broken stone)	30	——	311
1:2:5 "	90	2413	359
1:2:5 "	28	2290	237
1:5 (gravel)	90	2804	290
1:5 "	28	2400	253

Tests by Professor Ira H. Woolsen § on 1:2:4 mixtures 5 to 7 weeks old gave an average tensile strength of 161 lbs/in², compared to 1753 lbs/in² compressive strength.

Professor Talbot obtained values for 1:3:6 concrete from 50 to 84 days old of 178, 160, and 170 lbs/in².||

22. Tensile Strength as Determined by Transverse Tests.—The transverse strength of plain concrete depends almost entirely upon its tensile strength, although the modulus of rupture is considerably greater than the strength in plain ten-

* Bulletin No. 2, Vol. 4, Univ. of Wis., 1908.
† Jour. Assn. Eng. Soc., Sept. 1900.
‡ Jour. West. Soc. Eng., Vol. IX, 1904, p. 234.
§ Eng. News, Vol. LIII, 1905, p. 561.
|| Bulletin No. 1, Univ. of Ill., 1904.

sion owing to the curved form of the stress-strain diagram. Feret* found a very nearly constant ratio of 1.95 of modulus of rupture to tensile strength. The value of this ratio will ordinarily range from 1.8 to 2. Transverse tests of different concretes should therefore show about the same relative results as tensile tests. They are in fact quite as significant in this connection.

Some of the best tests on transverse strength are those made by William B. Fuller, and given in full in Taylor and Thompson's work on Concrete.† The following average results were obtained for 33–35-day tests.

Mixture by Volume.	Average Modulus of Rupture.
1:2.16:4.08	439 lbs/in²
1:2.16:5.1	380 "
1:3.24:5.1	285 "
1:3.24:6.12	226 "
1:3.24:7.14	239 "

Here we find the strength of the 1:3.24:6.12 mixture only about one-half that of the 1:2.16:4.08 mixture, indicating the relative weakness in tension of the lean mixture.

The results herein given, both of tensile and of transverse tests, indicate that the quality of the concrete has a greater relative effect on the tensile strength than on the compressive strength, the strength of a 1:3:6 mixture being not more than two-thirds that of a 1:2:4 mixture. Reasonable values for ultimate tensile strength would appear to be about as follows:

1:2:4 mixture.........	160–200 lbs/in²
1:3:6 "	100–125 "

23. Shearing Strength.—There is some confusion among writers as to just what is meant by the term "shearing strength," resulting in a wide variation in the suggested values for working stresses. In this work the authors will use the term to denote the strength of the material against a sliding failure when

* Etude Expérimentale du Ciment Armé. Paris, 1906.
† Concrete, Plain and Reinforced, N. Y., 1906.

tested as a rivet or bolt would be tested for shear; or the shearing of metal by punching a hole in a plate. This action is sometimes called "punching" shear.

Some writers used the term "shearing-stress" to mean quite a different thing from that described above, namely, the complex action which occurs in the web of a beam. In this case there exist direct tensile and compressive stresses which at the neutral axis are equal in intensity to the vertical and horizontal shearing stresses. The limit of distortion in the concrete will be reached, and failure will occur, when the tensile strength of the material is exceeded. Such a failure may perhaps be called a shearing failure, but is more strictly a failure in tension in a diagonal direction, and is so considered in this work. In practice the diagonal tensile stresses in a beam must often be considered, but shearing stresses, as such, will be dangerous only in exceptional circumstances, such as exist where a heavy load is applied close to a support.

Tests made under the direction of Professor C. M. Spofford on cylinders 5 inches in diameter with ends securely clamped in cylindrical bearings gave results as follows:

Mixture.	Shearing Strength, lbs/in².	Compressive Strength, lbs/in².	Ratio of Shearing to Comp. Strength.
1:2:4	1480	2350	.63
1:3:5	1180	1330	.89
1:3:6	1150	1110	1.04

Tests made at the University of Illinois on rectangular specimens tested in a similar manner gave the following average results:

Mixture.	Shearing Strength, lbs/in².	Compressive Strength, lbs/in².	Ratio of Shearing to Comp. Strength.
1:2:4	1418	3210	.44
1:3:6	1250	2290	.57

Tests made by punching through plates gave shearing strengths varying from 37 to 90% of the compressive, the value depending upon the form of test-piece.*

* Bulletin No. 8, Univ. of Ill., 1906.

Tests by M. Feret on mortar prisms gave results for shearing strength equal to about one-half the crushing strength.

The ordinary crushing failure is really a failure by shearing (combined with compression), and under such conditions the crushing stress is, theoretically, twice the shearing-stress, the angle of shear being 45°. Results of tests give a somewhat greater inclination than 45°, so that the crushing stress is somewhat greater than twice the actual shearing-stress.

We may then conclude, both from theory and from tests, that the shearing strength of concrete, in the sense here used, is nearly one-half the crushing strength. It is in fact so large that it will need to be considered only in exceptional cases.

24. Elastic Properties of Concrete.—*Stress-strain Curve in Compression.*—In the design of combination structures, such

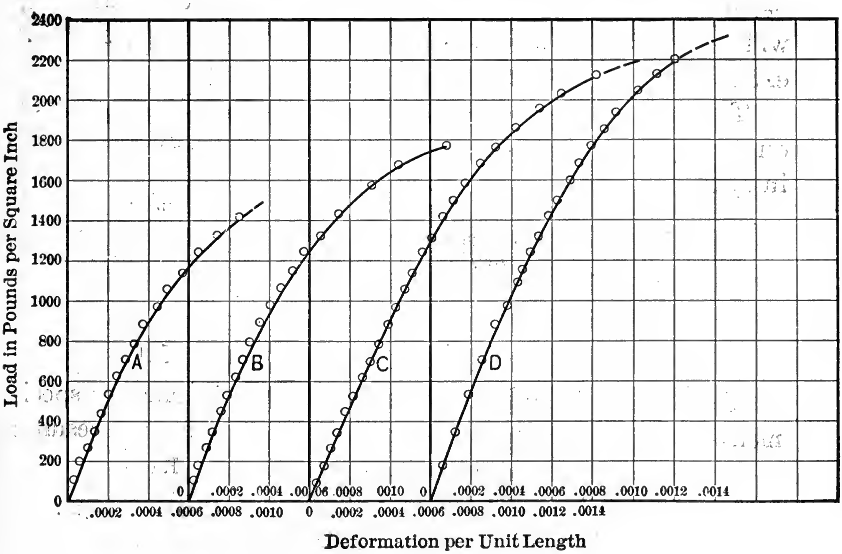


FIG. 3.—Stress-strain Diagrams in Compression.

as those of steel and concrete, it is necessary to know the relative stresses under like distortions. These will depend upon the moduli of elasticity of the two materials. For purposes of safe design we need to know also the elastic-limit strength.

Fig. 3 represents typical stress-strain curves for concrete in compression obtained from tests on cylinders 6 inches in diameter by 18 inches high. The concrete was 1 : 2 : 4 lime-

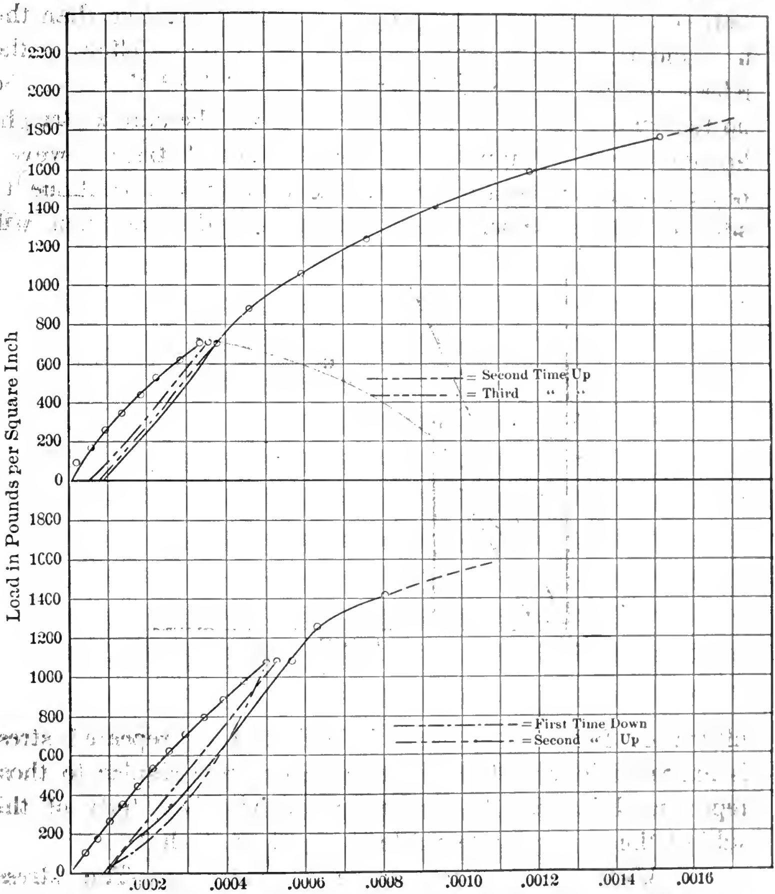


Fig. 4.—Stress-strain Diagram under Repeated Leads.

stone concrete 30 days old. The ultimate strengths ranged from 1500 to 2300 lbs/in².

Unlike the elastic line for steel, the line for concrete is slightly curved almost from the beginning, the curvature gradually increasing towards the end. There is no point of sharp

curvature as for ductile materials. A release of load at a moderate stress, such as 500 to 600 lbs/in^2, will usually show a small set indicating imperfect elasticity. A second application of the load will, however, give a straighter line than the first and there will be much less permanent set following the release of load. After a few repetitions of load there will be no further set and the stress-strain line will become a straight line up to the load applied. There is a limit of stress, however, beyond which repeated applications of load will continue to add to the permanent deformation and the specimen will

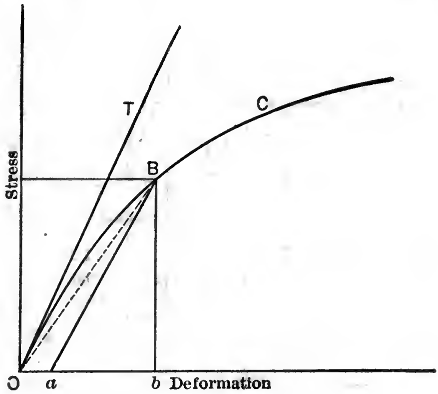


FIG. 5.

ultimately fail. The general behavior under repeated stress is indicated in Fig. 4, from tests on concrete similar to those represented in Fig. 3. For a very exhaustive study of this subject the reader is referred to the work of Bach.*

25. *Modulus of Elasticity in Compression.*—The stress-strain line being curved almost from the beginning, the proper method of calculating the modulus of elasticity needs to be considered. Fig. 5 is a typical stress-strain diagram for compression (somewhat simplified). At B the load has been removed and reapplied, giving the set Oa and the stress-strain line aB, an approximately straight line.

* Zeit. V. dt. Ing., 1895, etc.

For very low stresses, up to perhaps 300 to 400 lbs/in^2. (a low working stress), the variation of the curve from a straight line is so small that the modulus of elasticity may be taken as the slope of the tangent line at the origin OT. For higher stresses such as would correspond to a point B on the curve, the modulus of elasticity is not so definite, it depends upon what is meant by the term "modulus of elasticity," in such a case.

There being no general agreement as to the exact definition of the term for such materials as concrete, the method which should be employed in calculating its value should depend upon the purpose for which it is to be used. The principal use of the modulus of elasticity in reinforced-concrete design is to determine the relative stresses carried by the concrete and steel on the assumption that there is perfect adhesion between the two materials and, therefore, equal distortions. This being the case, the distortion which should be used in calculating the modulus for the concrete should be the total distortion from the beginning.

Consider, for example, the action in the case of a column. Assuming no initial stress in the steel or concrete, suppose that the column is loaded so as to cause a shortening equal to Ob, Fig. 5. The stress in the concrete will be bB, and that in the steel will be equal to the deformation Ob multiplied by its modulus of elasticity. Upon removal of the load there may be a permanent set Oa, which means that there is some residual compression in the steel (with an equal amount of tension in the concrete). A second application of the load will cause a deformation ab, but, measuring from the original position, the deformation is Ob, and this again fixes the stress in the steel. Hence, for the determination of the relative stresses in steel and concrete, the modulus for the concrete should be the ratio of Bb to Ob, or the slope of the chord OB.

In the case of a beam the stresses in the concrete at any section will vary from zero at the neutral axis to the value Bb, for example, at the extreme fibre. At intermediate points the stresses follow approximately the law of the curve OB.

In this case a chord *OB* does not exactly represent the facts, but the error is small, and it is the best line to use if the rectilinear variation of stress be assumed. If a curvilinear law is used, then the modulus should be the slope of the tangent at the origin. In neither case is it correct to use the slope of the line *aB*.

In referring to these various methods of calculating the modulus, the slope of the tangent *OT* is generally called the "initial modulus." The slope of the line *OB* is called the "secant modulus."

The value of the modulus for concrete varies greatly as determined by different experimenters and for different kinds of concrete. As a rule the denser and older the concrete the higher the modulus.

Among the most careful experiments are those by Bach,* in which he repeated the loads at each increment until there was practically no increase of set.

The following are some average results:

Kind of Concrete.	Modulus of Elasticity, lbs/in².		
	Based on Elastic Deformation.		Based on Total Deformation.
	At 114 lbs/in².	At 570 lbs/in².	At 570 lbs/in².
1:2½:5 (broken stone)........	4,660,000	3,590,000	3,440,000
1:2½:5 (gravel)..............	3,170,000	2,520,000	2,200,000
1:3:6 (broken stone)........	3,870,000	2,990,000	2,570,000
1:3:6 (gravel)..............	3,000,000	2,240,000	2,110,000

The specimens were 25 cm. in diameter and 100 cm. high and were from three to four months old.

Tests of cylinders one month old of 1 : 2 : 4 concrete by the U. S. Geological Survey,† gave average values of the initial modulus as follows:

* Zeit. V. dt. Ing., 1895.
† Bull. No. 344, p. 52.

Consistency.	Nature of Coarse Aggregate.		
	Granite.	Gravel.	Limestone.
Wet............	3,570,000	3,790,000	3,590,000
Medium.........	4,080,000	3,880,000	3,430,000
Damp..........	4,890,000	4,070,000	4,260,000

The wet consistency was "mushy" and splashed under the action of the tamper.

Considering the various results obtained and the significance of total deformation it would appear that for working loads the modulus for ordinary concrete ranges from 2,500,000 to 3,500,000 lbs/in², depending upon the mixture and the age of the concrete. As will be shown subsequently, however, the value selected should also depend upon the purpose for which it is to be used, and that for most calculations relating to strength a value of 2,000,000 for ordinary concrete is more satisfactory than a higher value.

26. *Poisson's Ratio.*—When a material is subjected to compressive stress a certain amount of lateral expansion takes place. The ratio of such lateral expansion to the longitudinal compression is known as *Poisson's ratio.* In concrete this ratio varies generally from one-sixth to one-twelfth. Talbot found values from 0.1 to 0.16 for working loads for 1 : 2 : 4 concrete at sixty days.* Withey found the following values for 60-day concrete for loads equal to one-fourth the ultimate: for 1 : 3 : 6 mix, 0.08; for 1 : 2 : 4 mix, 0.11; for 1 : 1⅓ mortar, 0.16.†

27. *Elastic Limit.*—As stated in the preceding article, concrete shows a permanent set under small loads so that, in the usual sense, the material can hardly be said to have an elastic limit. There appears to be, however, a limit to the stress which can be repeated indefinitely without continuing to add to the deformation, and this limit may be taken as the elastic limit for practical purposes. From experiments by Bach and

* Bull. No. 20 Univ. of Ill.

† Bull. No. 466 Univ. of Wis.

others, this limit seems to be from one-half to two-thirds the ultimate strength. In repeated-load experiments on neat cement and on concrete made by Professor J. L. Van Ornum * it has been shown that the maximum load which may be repeated an indefinite number of times without rupture does not much exceed 75% of the ultimate strength. These results show a close relation to those obtained by Bach, and it may therefore be concluded that the limit of permanent elasticity for repeated loads is from 50 to 60% of the ultimate strength.

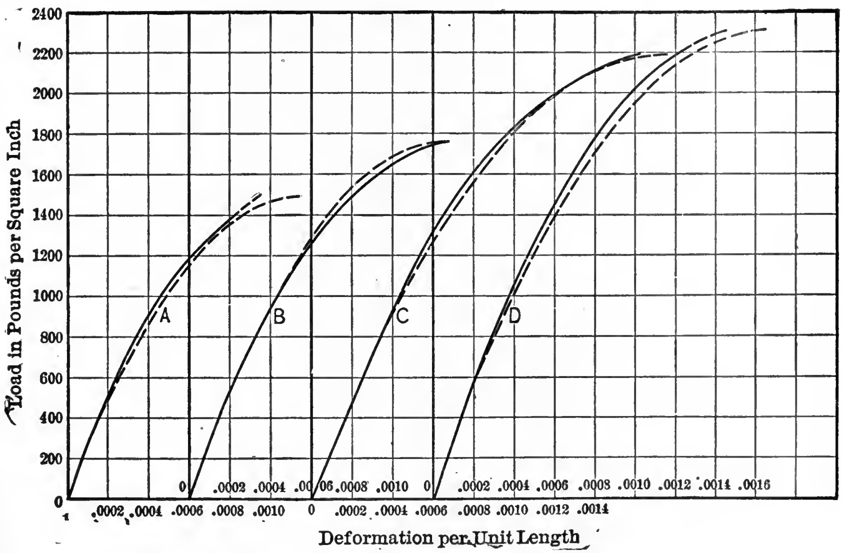


FIG. 6.

The property of concrete of undergoing a certain amount of permanent deformation at relatively low loads, without injury, may be thought of as a kind of "plasticity." This "plasticity" is of some advantage in enabling the concrete in certain details of construction to adjust itself somewhat to the deformations without a corresponding increase of permanent stress.

* Trans. Am. Soc. C.E., Vol. LI, p. 443.

28. *Comparison of Stress-strain Curve with the Parabola.*—As the parabola is often used in theoretical analyses to represent the stress-strain curve it will be useful to compare some typical curves with the parabola. The form of parabola used has its axis vertical and its vertex at the point of the curve representing the ultimate strength. In Fig. 6 the curves shown in Fig. 3 are compared with parabolas (shown in dotted lines). The agreement is seen to be very close.

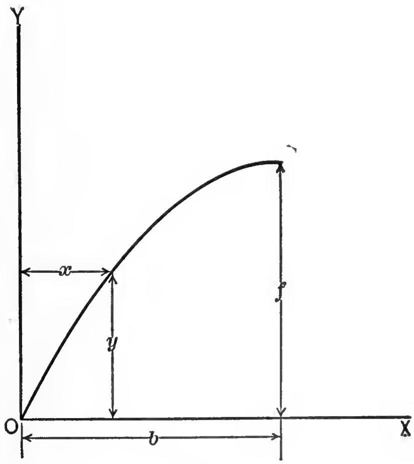


FIG. 7.

If f=stress at rupture and b=ultimate deformation, the equation of the parabola is (Fig. 7):

$$\frac{f-y}{f}=\frac{(b-x)^2}{b^2},$$

or

$$y=f-\left(\frac{b-x}{b}\right)^2 f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

29. *Stress-strain Curve for Tension.*—Comparatively few tests have been made on the elasticity of concrete in tension. Bach found for 1 : 4 concrete an average value of the modulus of 3,800,000 lbs/in^2 at a stress of 80 lbs/in^2, and 3,100,000

at a stress of 135 lbs/in². The ultimate tensile strength was 185 lbs/in². The modulus in compression for the same concrete was 3,850,000 at 80 lbs/in².* Professor Hatt has determined values ranging from 2,000,000 to 5,000,000 lbs/in²,

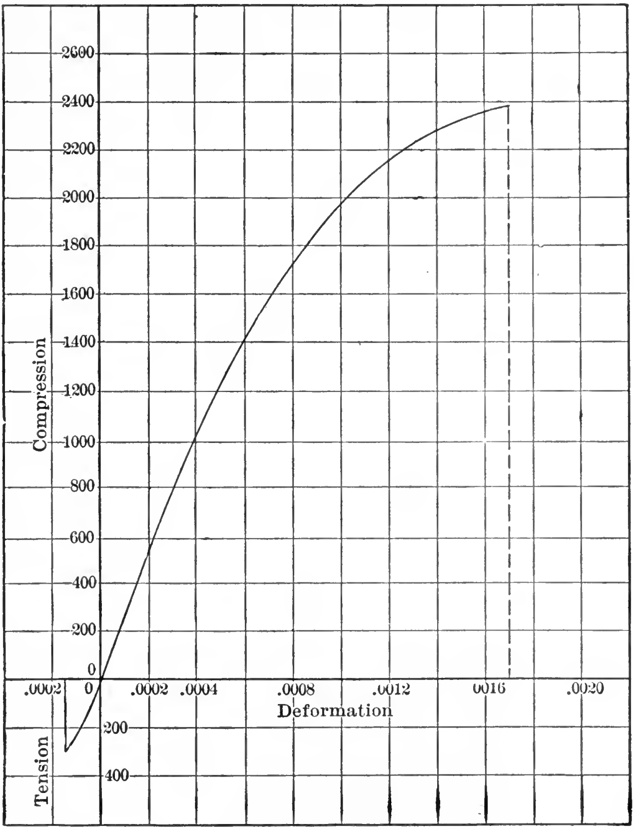


Fig. 8.—Relative Strength and Deformation in Compression and Tension.

which were generally about equal to the values in compression.† These and other tests indicate that the initial moduli in tension and in compression are about the same, and as the working

* Mitt. Ueber Forsch. a. d. Gebiet des Ing., 1907, Heft 45–47

† Proc. Am. Soc. Test Mat., 1902.

limit in tension is very low they may be assumed as equal. The relative strength and deformation of concrete in compression and tension is illustrated by a typical curve in Fig. 8.

30. Coefficient of Expansion.—Experiments by Professor W. D. Pence * on 1 : 2 : 4 concrete gave an average value of the coefficient of expansion of .0000055 per degree Fahrenheit, there being little variation among the several tests. Tests made at Columbia University on 1 : 3 : 6 concrete gave values of about .0000065. Other experiments have shown somewhat higher results A value of .000006 may be assumed.

31. Contraction of Concrete in Hardening and Drying.—When concrete hardens in air it shrinks and when hardened in water it usually expands slightly. The shrinking appears to be due to the drying-out process rather than the hardening process itself, as it is found that nearly the same shrinkage will occur when concrete hardened in water is dried out. Goldbeck found that 1 : 2 : 4 dry-mixed concrete showed a contraction of .034% in thirty days and .048% in sixty days, when exposed to air, and wet-mixed concrete .011% and .041%, respectively. When hardened under moistened burlap for fifteen days there was at first a slight expansion and then on free exposure to the air a contraction in sixty days of .016% at .025%, depending upon the consistency of the mix.

Generally speaking, it may be said that the change of length of concrete from a state of saturation in water to a state of thorough dryness is comparable to that produced by a change of temperature of about 50 to 60 degrees.

32. Weight of Concrete.—The weight of concrete of the usual proportions will vary from 140 to 150 lbs/ft^3, depending upon the degree of compactness and the specific gravity of the materials. Variation of proportions will affect the weight but little if the proper ratio of sand and stone be maintained, but a wet concrete when dried out will weigh less than a well-compacted concrete containing originally less water. For prac-

* Jour. West Soc. Eng., Vol. VI, 1901, p. 549.

tical purposes an average value of 145 lbs/ft³ may be taken. The addition of reinforcing steel in the usual proportions will add from 3 to 5 pounds, so that the weight of reinforced concrete may be taken at 150 lbs/ft³.

33. Properties of Cinder Concrete.—The following table of results indicates fairly well the strength and modulus of elasticity of cinder concrete. The age of the specimens varied from 30 to 100 days. Cinder concrete will weigh from 110 to 115 lbs/ft³.

TABLE NO. 5.

CRUSHING STRENGTH AND MODULUS OF ELASTICITY OF CINDER CONCRETE.

WATERTOWN ARSENAL TESTS, 1898.

Mixture.			Average Crushing Strength, lbs /in ².		Average Modulus of Elasticity between Loads of 100 and 600 lbs /in ².
Cement.	Sand.	Cinders.	One Month.	Three Months.	
1	1	3	1540	2050	2,540,000
1	2	3	1098	1634	
1	2	4	904	1325	
1	2	5	724	1094	1,040,000
1	3	6	529	788	

REINFORCING STEEL.

34. General Requirements.—In general, reinforcing steel must be of such form and size as to be readily incorporated into the concrete so as to make a monolithic structure. To provide the necessary bond strength and to distribute the steel where needed without concentrating the stresses on the concrete too greatly, requires the use of the steel in comparatively small sections. This requirement, as well as that of economy and convenience, leads to the use of the steel in the form of rods or bars. These will vary in size from about $\frac{1}{4}$ to $\frac{3}{8}$ inch for light floors up to $1\frac{1}{2}$ to 2 inches as maximum sizes for heavy beams or columns. Under certain conditions

a riveted skeleton work is preferred for the steel reinforcement, but this is usually where for some reason it is desired to have the steelwork self-supporting or where it is to carry an unusually large proportion of the load.

35. Forms of Bars.—Plain round and square rods are largely used, the adhesion of the steel and concrete being depended upon to furnish the necessary bond strength. Plain flat bars are undesirable unless used in connection with riveted reinforcement, as their adhesion to the concrete is much less than that of round or square bars. Many special forms of bars have been devised, the principal object of which is to furnish a bond with the concrete independent of adhesion,—a mechanical bond, as it is usually called. Some of the most common types of such bars are illustrated in Fig. 9. Many other devices are employed to a greater or less extent to provide a mechanical bond, and numerous combinations of forms are used as patented "systems" in the construction of beams, floors, and columns.

36. Quality of Steel.—Steel bars used in reinforced work are not usually subjected to as severe treatment as those used in ordinary structural work. They must be capable of being bent to the desired form, but this is the only treatment to which the ordinary bars are subjected. In many concrete structures the impact effect is also likely to be less than in all-steel structures; consequently it is considered that a somewhat less ductile material may safely be used. These conditions lead to the use, by many engineers, of high elastic limit material, while others, probably the majority, prefer the use of standard structural steel.

High elastic-limit bars made by re-rolling rails have been used extensively, but such steel is less reliable as to composition than new material, and is not recommended by the Joint Committee. However, the American Society for Testing Materials has adopted standard specifications for both billet steel and rail steel. The specifications for billet steel provide for three grades—structural, intermediate, and hard—to be either Bes-

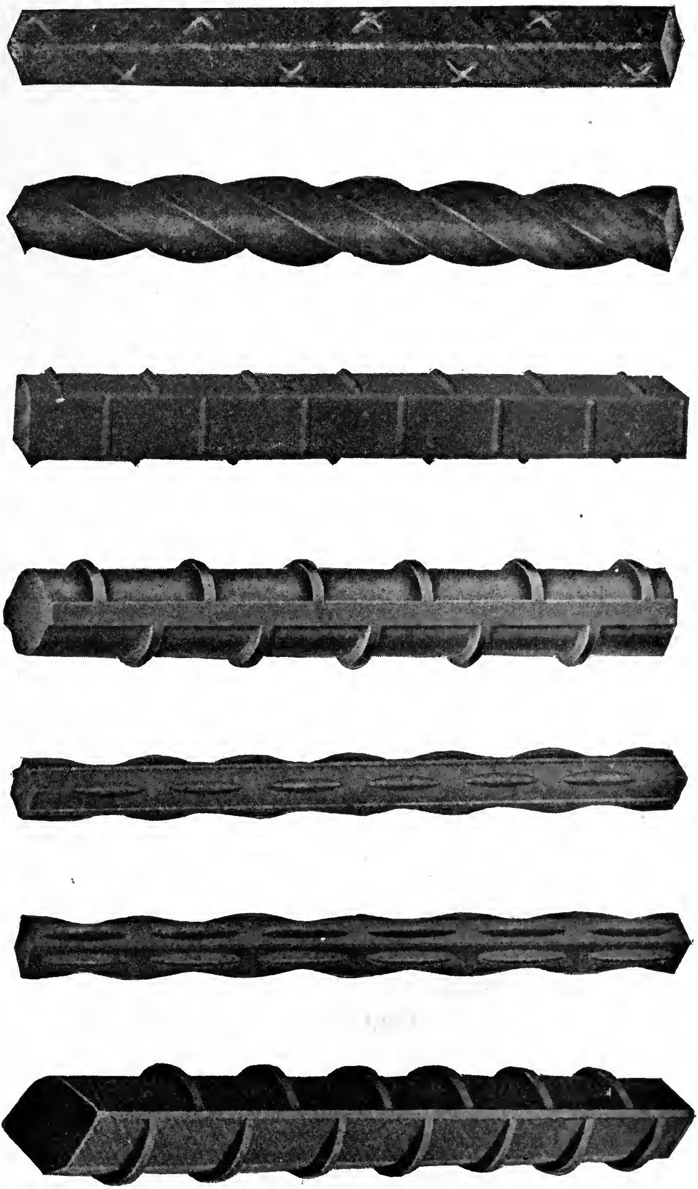


Fig. 9.—Types of Deformed Bars.

semer or open hearth, and having ultimate strengths as follows:

Structural	55,000 to 70,000 lbs/in²
Intermediate..............	70,000 to 85,000 lbs/in²
Hard......................	80,000 lbs/in² and above

The structural grade is the grade recommended by the Joint Committee. It corresponds closely to standard bridge material in ultimate strength, but the requirements as to elongation and bend test are less severe. The specifications for rail steel prescribe a minimum tensile strength of 80,000 lbs/in². The requirements for deformed bars are the same as for plain bars except as to elongation and bending, in which respect the requirements are somewhat lower.

37. Modulus of Elasticity.—The modulus of elasticity of all grades of steel is very nearly the same and will be taken at 30,000,000 lbs/in².

38. Coefficient of Expansion.—The coefficient of expansion of steel may be taken at .0000065 per 1° F.

PROPERTIES OF CONCRETE AND STEEL IN COMBINATION.

39. Adhesion or Bond Strength.—The high value of the tangential adhesion, or bond, of concrete to steel rods embedded therein had been known long before the use of reinforced concrete, and had been utilized in the placing of anchor rods, etc. It was, in fact, somewhat remarkable that this property was not earlier made use of in the design of combination structural forms. Experience has shown that the bond resistance between concrete and plain round and square bars is sufficiently reliable and permanent to enable them to be used in permanent structures and the use of the plain bar is the general practice in Europe. Bars of irregular section in which adhesion is not entirely depended upon for the bond are also used to a large extent. Some form of mechanical bond is necessary where the adhesion area is deficient, and some engineers consider such a bond desirable in all cases.

The value of the bond resistance depends very largely upon the character of the concrete and the smoothness of the surface of the bar. It also depends upon the nature of the stresses and deformations that exist in the concrete and steel. Thus the bond strength shown by a test on a beam is different from that shown by pulling a bar from a test-block. Since a careful study of this question requires some knowledge of the stresses in reinforced concrete beams, the detailed discussion of the subject will be deferred to a subsequent chapter. It will suffice to state here that for plain round or square rods the ultimate bond strength may be taken at from 200 to 300 lbs/in². For deformed rods the resistance to initial slip is about the same as for plain rods, but the ultimate strength is much greater.

40. Ratio of Moduli of Elasticity, $E_s/E_c=n$.—So long as the adhesion between steel and concrete is unimpaired the distortion of the two materials will be equal. Their stresses will then be proportional to their moduli of elasticity for the load in question, or as the ratio of $E_s : E_c$. Taking E_s at 30,000,000 and E_c at from 2,000,000 to 3,000,000, the ratio will vary from 15 to 10. In practice various values of this ratio are used, depending upon the kind of concrete and the judgment of the designer. For the ordinary concrete this ratio is usually taken at 15, which is the value specified in the recommendations of the Joint Committee for concrete having a compressive strength between 800 and 2200 lbs/in².

41. Tensile Strength and Elongation of Concrete when Reinforced.—We have seen that plain concrete has an ultimate tensile strength of about 200 lbs/in² and a total elongation of perhaps $^1/_{10000}$ part, corresponding to a value of 2,000,000 for E_c. Steel stretches this amount under a stress of $30{,}000{,}000/10{,}000=3000$ lbs/in². Again, the safe working tensile stress of concrete is about 50 lbs/in², and if we use a value of $E_s/E_c=15$, the corresponding stress in the steel will be but 750 lbs/in². From these relations it is evident that in reinforced tension members we must either use very low and uneconomical working stresses for steel, or else expect

the concrete to crack and to be of no assistance in carrying stress.

Some early experiments by Considère on reinforced concrete beams seemed to indicate that concrete on the tension side would elongate before cracking much more than in an ordinary tension test, and that it could, therefore, be counted upon to carry a certain amount of tension under working conditions. Later experiments made at the University of Wisconsin * failed to confirm these results. On the contrary they showed that the concrete cracks at about the same deformation

TABLE No. 6.

TESTS OF BEAMS SHOWING EXTENSIBILITY OF CONCRETE.

No.	Age.	Method of Loading.	Proportionate Extension at First Crack.	Tensile Stress at First Crack, lbs/in².	Compressive Strength of Cubes, lbs/in².
8	3 months	At third points	.00011	220	4250
10	"	"	.00024	440	2500
26	"	"	.00016	320	3000
30	"	"	.00012	240	2600
7	1 month	At Center	.00015	300	3500
13	"	"	.00009	180	2350
23	"	"	.00020	400	2500
			At rupture		
2 [1]	"	"	.00013	260	3000
1 [1]	"	"	.00010	200	2500

[1] Nos. 2 and 1 were plain concrete beams. The extensions of the beams loaded at the third points were measured by extensometers; those of the center-loaded beams were calculated from deflections.

as in a tension test. However, on account of the presence of the steel, the cracks open up very slowly so that they are at first very difficult to detect.

The preceding table shows the results of some of the experiments at the University of Wisconsin, showing the deformation at the first indication of cracking, and the calculated tensile strength, assuming $E_c = 2{,}000{,}000$. The beams were of 1 : 2 : 4 mix and were 6×6 inches in cross-section by 60 inches span length.

* Bull. No. 4, Engineering Series, 1906.

It will be noted that the plain concrete beams, Nos. 1 and 2 of the table, show about the same deformations before rupture as the reinforced beams. These results have been confirmed by numerous experiments by Bach, who concludes that reinforced concrete will begin to crack at the same elongation as plain concrete. The calculated tensile strength of the concrete at first crack in the case of reinforced beams he determined to be about 1.4 times the strength in a straight tension test and the same as in reinforced beams.*

The presence of these minute cracks of course seriously affects the tensile strength of the concrete, and as they appear at an elongation corresponding to a stress in the steel of 5000 lbs/in^2 or less, it is evident that no allowance should be made for the tensile resistance of the concrete where the usual working stresses are used for steel. In some cases, however, the stresses in the steel are necessarily very low, in which case it may be proper to consider the tensile resistance of the concrete. This limit may be placed at about 2000 lbs/in^2, corresponding to an elongation of .00006 part and a stress of 150 to 175 lbs/in^2 in the concrete.

In practical design a more important question is how far a concrete may be cracked without exposing the steel to corrosive influences. In this respect experience indicates that the minute cracks which appear under working loads are of no practical consequence.

42. Contraction and Expansion of Reinforced Concrete.— The behavior of reinforced concrete as regards contraction and expansion, and the stresses resulting therefrom, may be considered under two conditions: (1) when the structure or part under consideration is not restrained by surrounding structures, and is, therefore, free to contract and expand as a whole; (2) when the part under consideration is restrained so that contraction and expansion are prevented.

1. When the structure is not restrained, then the only

* Heft 24, 1913, Deutscher Ausschuss für Eisenbeton.

stresses will be those resulting from a difference in deformation of the concrete and steel. Temperature changes affect the steel and concrete nearly alike (Arts. 30 and 38), so that the two materials will be but slightly stressed by reason of temperature changes.

The effect of shrinkage in hardening or drying out is more serious. As shown in Art. 31, the hardening of concrete is accompanied by contraction if in air, or expansion (to a less degree) if in water. Concrete which is unrestrained either by steel reinforcement or by exterior attachment will shrink or swell proportionally and no stresses will thereby be developed. If restrained by reinforcing material only, a shrinkage will develop tensile stresses in the concrete and compressive stresses in the steel.

If it be assumed that concrete when reinforced tends to shrink the same amount as plain concrete, and that such shrinkage is prevented only so far as the stresses developed in the steel react upon the concrete and cause an opposite movement, then it will be found, using the ordinary values of the modulus of elasticity, that the stresses developed in both the concrete and the steel will be large. These stresses may be estimated as follows:

Let c = coefficient of contraction of the concrete;
f_c = unit stress in concrete (tensile);
f_s = unit stress in steel (compressive);
p = steel ratio;
$n = E_s/E_c$.

Then the net contraction per unit length as measured by the concrete will be $c - f_c/E_c$, and as measured by the steel will be f_s/E_s. These values are equal. Also, for equilibrium, $f_c = pf_s$. From these equations we get

$$f_c = cE_c\frac{np}{1+np} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$f_s = \frac{f_c}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If, for example, $c=.0003$, $E_c=2{,}000{,}000$, $n=15$, $p=1\%$, then $f_c=80$ lbs/in² tension and $f_s=8000$ lbs/in² compression. If $p=2\%$, $f_c=140$ and $f_s=7000$ lbs/in².

It is doubtful if such large initial stresses actually occur in reinforced concrete due to shrinkage in hardening. In slowly hardening, with the steel in place, there is probably a gradual adjustment in the concrete which results in less internal stress than the experiments on plain concrete would indicate.

2. When the structure is restrained by outside forces so that it is not free to contract or expand, as in the case of a long wall, then the resulting stresses are likely to be high. If it be assumed that concrete when reinforced will not stretch more than plain concrete, as seems well proven (Art. 41), then no amount of reinforcement can entirely prevent contraction cracks. The reinforcement can, however, force such cracks to take place, as they do in a beam, at such frequent intervals that the requisite deformation is provided without any one crack becoming large. Laboratory tests on beams would indicate that if steel is used in sufficient quantities the cracks may easily remain quite invisible and be of no consequence from any practical standpoint. With a coefficient of expansion of .000006 a temperature change of 50 degrees will cause a change of length (if free) of .0003 part, about the same amount as the shrinkage in hardening and drying out. A deformation of .0003 part corresponds to a stress of about 600 lbs/in² in the concrete, a stress much beyond the ultimate tensile strength. Hence temperature changes and shrinkage are quite certain to cause cracks; but if the concrete is well reinforced such cracks may be kept very small. For example, a deformation of .0003 part on the tension side of a beam corresponds to a stress of about 9000 lbs/in² in the steel, which is much below the usual working stress and which would not cause cracks easily detected. The prevention of large cracks by means of reinforcement is, then, a matter of using sufficient steel to force the concrete to crack at small intervals. No crack will open up far until the steel is stressed beyond its elastic limit, hence we may say that the

amount of steel should be at least enough to force the concrete to crack at a second point before the steel reaches its elastic limit; that is to say, the elastic limit strength of the steel should be greater than the tensile strength of the concrete. A still larger amount of steel will serve to keep the cracks smaller.

The size and distribution of the cracks will also depend upon the bond strength furnished by the rods. If we assume the cracks to develop successively the distance between cracks must be sufficient to develop a bond strength equal to the tensile strength of the concrete. Hence, in general, the size and spacing of the cracks will vary inversely with the bond strength of the reinforcing steel per unit of concrete section.

For reinforcement against shrinkage and temperature stresses, a high elastic-limit steel is desirable, and in order to distribute the deformation as much as possible a mechanical bond is advantageous. The amount of steel necessary for such reinforcement depends upon the thickness and exposure of the structure. For thin walls and exposed locations 0.4% to 0.5% is required, while under very favorable conditions as little as 0.1% has been found to be sufficient. The reinforcement for this purpose should be placed close to the exposed faces of the concrete. In floor slabs longitudinal bars of $\frac{1}{4}$ inch to $\frac{1}{2}$ inch diameter, spaced about 2 feet apart, are customary.

CHAPTER III.

THEORY OF THE FLEXURE OF BEAMS.

43. Kinds of Members.—Structural members are, for convenience, usually divided into *tension members, compression members,* and *beams,* according as the forces to be resisted produce in the member simple tension, simple compression, or simple bending. Bending moment is often accompanied by tension or compression, producing what are called *combined stresses* of *bending and tension,* or *bending and compression.* Since reinforced concrete is not used for plain tension members the analysis will be confined to the beam, both under plain bending and under combined stresses, and to the compression member or column. The flat slab supported on four sides or by columns at four points, will be considered in a separate chapter as a special case of beam. In reinforced-concrete construction the beam is the most important element being used under a great variety of conditions.

44. Relation of Stress Intensities in Concrete and Steel.—In the following discussion it will be assumed that the concrete and steel adhere perfectly and therefore deform equally. Nearly all reinforced-concrete construction is dependent upon this equal action of the two materials, although simple adhesion is not always entirely depended upon. Many types of deformed, or roughened, bars are used so as to give the steel a grip independent of the adhesion, and in other cases bars are bent or anchored at the ends, but in all cases it is assumed that the materials adhere perfectly and therefore deform equally. Many tests show that under proper design this is for all practical purposes true.

Since the modulus of elasticity of a material is the ratio of stress to deformation, it follows that for *equal* deformations

the stresses in different materials will be as their moduli of elasticity. If

f_s = unit stress in steel,
f_c = unit stress in concrete,
E_s = modulus of elasticity of steel, and
E_c = modulus of elasticity of concrete,

we have the fixed relation

$$f_s/f_c = E_s/E_c.$$

45. Distribution of Stress in a Homogeneous Beam.—To assist in forming correct notions of the action of steel reinforce-

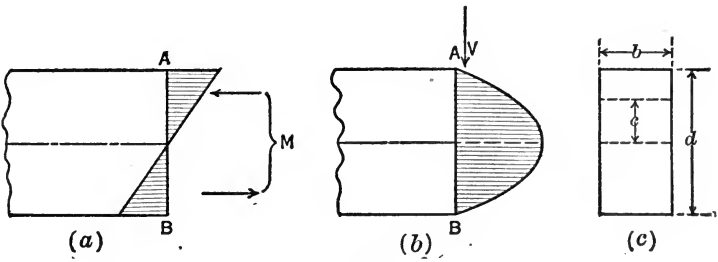


FIG. 1.

ment in a concrete beam, it will be desirable to consider, at the outset, the nature of the stresses due to bending moment in a plain concrete beam or a beam of any homogeneous material.

Considering a vertical section at any point of the beam, Fig. 1, there will exist in general certain tensile and compressive stresses acting normally to the section and certain vertical or shearing stresses acting tangentially thereto. In accordance with the common theory of flexure, the normal stress on a vertical section varies in intensity as the distance from the neutral axis, and therefore the variation is represented by the ordinates to a straight line as in Fig. 1 (*a*). The intensity of this stress (fibre stress) at any point is given by the well-known formula

$$f = \frac{Mc}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which M = bending moment, c = distance of fibre from neutral axis, and I = moment of inertia of the section with respect to the neutral axis.

The shearing-stress intensity is a maximum at the neutral axis and is zero at the outer fibres. At any given point in the section it is given by the equation

$$v = VS/Ib, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which V = total vertical shear at the section through the point under consideration, b = breadth of the section at the given point, and S = statical moment of that part of the section above (outside of) the point with respect to the neutral axis. For a rectangular beam the intensity of shear varies as the ordinates to a parabola, as shown in Fig. 1 (*b*), the maximum value being 3/2 times the average, or equal to $\frac{3}{2} \cdot \frac{V}{bd}$. The intensity of the horizontal shearing stress at any point is equal to that of the vertical shearing stress.

A determination of the normal stresses (often called the "direct" or "bending" stresses) and the shearing stresses above

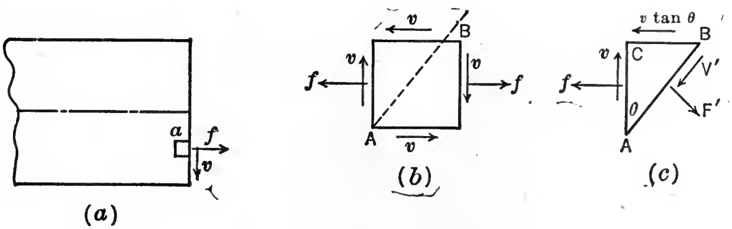


FIG. 2.

described gives sufficient information for the design of ordinary beams of homogeneous material. For purposes of reinforced concrete design, however, it is desirable to make a more detailed analysis of the stresses acting at any point in a beam.

Consider an element a at any point of a beam, Fig. 2 (*a*), of unit dimensions, and with horizontal and vertical faces.

Fig. 2 (*b*) is an enlarged view of this element. The direct and shearing-unit stresses may be represented by f and v respectively; their magnitudes will be determined by the formulas given above. Let us examine the stress on an inclined section AB. Fig. 2 (*c*) shows the portion on the left of the section. The stresses on the face AB may be resolved into a tension F' and a shear V'. The shear on CB will be $v \tan \theta$. Taking components normal to AB, we find $F' = (f + v \tan \theta) \cos \theta + v \sin \theta$ and $V' = v \cos \theta - (f + v \tan \theta) \sin \theta$.

The stresses per unit of area will be

$$f' = F' \cos \theta = f \cos^2 \theta + 2v \sin \theta \cos \theta;$$

$$v' = V' \cos \theta = v(\cos^2 \theta - \sin^2 \theta) - f \sin \theta \cos \theta.$$

For a maximum value of f' we find by differentiation,

$$\tan 2\theta = \frac{2v}{f}, \quad \ldots \ldots \ldots \quad (3)$$

and for maximum v'

$$\tan 2\theta = \frac{f}{2v}. \quad \ldots \ldots \ldots \quad (4)$$

Substituting these values of θ in the expressions for f' and v', we find the maximum values to be respectively

$$f'_{\max} = \tfrac{1}{2} f + \sqrt{\tfrac{1}{4} f^2 + v^2}, \quad \ldots \ldots \quad (5)$$

$$v'_{\max} = \sqrt{\tfrac{1}{4} f^2 + v^2}. \quad \ldots \ldots \ldots \quad (6)$$

These equations show that the maximum tensile stress in a beam depends upon both f and v at the point in question.

In all points in a beam where the shear is zero, the direction of the maximum tension is horizontal, as at points of maximum bending moment and along the outer fibres of the beam. Wherever the horizontal fibre stress f is zero (at the neutral surface and at all sections of zero bending moment), the direc-

tion of the maximum tension is inclined 45° to the horizontal, and its intensity is equal to the unit shearing stress at the same place. Above the neutral axis the inclination of the maximum tension is in general greater than 45°, becoming 90° at the upper or compressive fibre. Fig. 3 illustrates the vari-

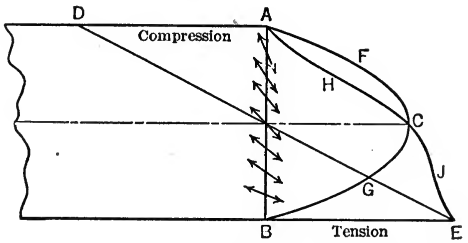


FIG. 3.—Variation in Maximum Tensile Stress.

ation in normal stress, shearing-stress, and maximum tensile stress throughout the entire depth of a rectangular beam. The outer normal or fibre stress is assumed at 200 lbs/in², and the shearing stress at the neutral axis at 150 lbs/in². The variation in the fibre stress is shown by the straight line *DE*, and that in the shearing stress by the parabolic curve *ACB*. By means of eq. (5) the maximum tensile stresses have been computed; these are represented by the line *AHCJE*.

The direction of the maximum tension is shown by the arrows along the line *AB*. The direction of the maximum compression is at right angles to that of maximum tension.

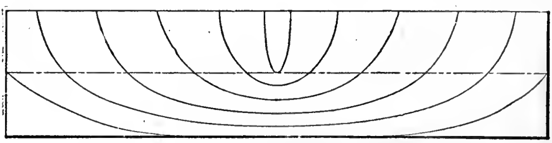


FIG. 4.—Lines of Maximum Tension.

Fig. 4 illustrates the direction of the maximum tensile stresses in a rectangular beam. The exact direction at any point depends upon the relation between shear and bending

moment. Lines of maximum compression would run at right angles to the lines shown and lines of maximum shear at angles of 45° therewith.

46. Purpose and Arrangement of Steel Reinforcement.—The purpose of steel reinforcement is to carry the principal tensile stresses, the concrete being depended upon for the compressive and shearing stresses, its resistance to such stresses being large. If no steel were present the concrete would tend to rupture on lines perpendicular to the direction of maximum tension, as shown in Fig. 4, and hence we may conclude that the ideal tension reinforcement would require the steel to be distributed in the beam along the lines of maximum tension. At the centre of the beam, or section of maximum moment, this direction is horizontal for the entire depth of the beam, and horizontal rods placed near the lower edge of the beam constitute proper and sufficient reinforcement. As we approach the ends of the beam, where the shear is large, the intensity of the inclined tensile stresses becomes of importance, and in many cases these stresses require special attention. Horizontal rods at the bottom are still necessary, but do not entirely reinforce the concrete against tension, so that special consideration must be given to reinforcement in the body of the beam. The arrangement of this reinforcement demands careful consideration.

For purposes of discussion, the subject of beams will first be treated with reference only to the horizontal reinforcement. The inclined tensile stresses will be considered separately under Shear, Chap. IV.

47. The Common Theory of Flexure and Its Limitations for Materials Like Concrete.—The common theory of flexure is based on two main assumptions, namely (1), a plane cross-section of an unloaded beam will still be plane after bending (Navier's hypothesis); (2) the material of the beam obeys Hooke's law, which is, briefly stated, "stress is proportional to strain." From the first assumption it follows that—*The unit deformations of the fibres at any section of a beam are pro-*

portional to their distances from the neutral surface. In the case of simple bending (all forces at right angles to the beam) the neutral axis lies at the centre of gravity of the section; in the case of bending combined with direct tension or compression, the neutral axis may lie in the section or be merely an imaginary line without the section. From the second assumption it follows that—*The unit stresses in the fibres of any section of a beam also are proportional to the distance of the fibres from the neutral surface.* This may be called the linear law of the distribution of stress.

While these two assumptions are commonly made in the analysis of a beam, they are not in all cases strictly correct, and in some cases Hooke's law cannot be used without large error.

The assumption of plane sections is slightly in error wherever shearing-stresses exist, as these tend to distort a plane section into one slightly S-shaped in form. The resulting error is, however, small and of no practical consequence, especially in concrete, where the effect is much less than in steel. Experiments by Talbot * failed to show any shearing effect in reinforced-concrete beams, but experiments by Schüle † seemed to show such distortion. In any case the variation from plane sections is negligible and hence it may be assumed that deformations of fibres will be proportional to distances from neutral surface.

The assumption that fibre stresses are proportional to deformations (Hooke's law) is practically correct for wrought iron and steel within the elastic limits, and hence the common theory of flexure gives correct results for these materials for working conditions; but for stresses beyond the elastic limit it does not apply. Other materials like timber, stone, cast iron and concrete do not obey Hooke's law so closely as steel and wrought iron, but for working stresses the variation

* Univ. of Ill. Bull., Vol. II, No. 1, p. 28.

† Mitteilungen der Materialprüfungs-Anstalt am Polytechnikum in Zürich, Vol. X (1906), p. 40.

is not great and the common theory may be used with closely approximate results. If a more exact analysis is desired for such materials the linear law must be discarded and the actual stress-strain diagrams in compression and tension must be used up to the limit of the actual stresses involved, or some form of mathematical curve approximating the true diagram.

48. Analysis of Beams from Stress-Strain Diagrams.—Where Hooke's law cannot be used, the true variation of the normal stress on the cross-section of a beam can be represented graphically as follows: Let Fig. 5 (*a*) be the actual stress-strain diagram

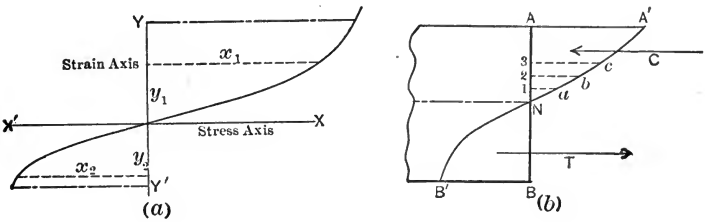


FIG. 5.

of the material in question, as determined by direct compression and tension tests, the curve above the axis representing the compression diagram and that below the axis the tension diagram. These curves are plotted with unit stresses as abscissas and unit strains as ordinates. Let Fig. 5 (*b*) represent a beam, cut on section *AB* where the stresses are to be investigated. Suppose the neutral axis is at *N* (the determination of its position is discussed later). Suppose, further, that the bending moment on the beam is such that the compressive stress on the top fibre at *A* reaches the value shown by abscissa x_1 on the stress-strain diagram. Lay off AA' equal to x_1. Then the distance AN may be considered as representing to some scale the deformation corresponding to the stress AA', or the deformation y_1. Then, since deformations are proportional to distances from N, we may lay off to the same scale other deformations $N1$, $N2$, etc., and from Fig. 5 (*a*) find the corresponding stresses and lay these off, $1a$, $2b$, etc. The

curve NA' then represents the variation of fibre stress in the beam. Similarly on the tension side the curve NB' is constructed, the same scale of deformation being used as for the upper part. The extreme deformation NB is found to be equal to y_2, in Fig. 5 (*a*) and the maximum stress BB' is equal to x_2. The diagram $AA'NBB'$ is evidently merely a portion of the stress-strain diagram of Fig. 5 (*a*) plotted to a different scale.

49. *Application to a Rectangular Concrete Beam.*—In the preceding article it was shown how the variation of the stress intensity in any beam might be determined from the stress-strain diagrams of the material, the neutral axis being known and the value of the stress at some fibre. It will now be shown how the neutral axis may be found and the actual moment of resistance determined for any given extreme fibre stress for the case of a rectangular beam.

For use in this and other discussions on flexure three important principles from the mechanics of beams are now recalled:

(1) For beams rectangular in section, the average unit tensile and compressive fibre stresses on any cross-section are represented by the average abscissas in the tensile and compressive parts of the stress diagram, NBB' and NAA', respectively (Fig. 5). Also the whole tension T and whole compression C on the cross-section are proportional to the *areas* NBB' and NAA'; hence, according to some scale, the areas represent T and C respectively.

(2) The resultant tension T and resultant compression C act through the centroids of the tensile and compressive areas in the stress diagram.

(3) When all the forces (loads and reactions) applied to the beam act at right angles to it, then the resultant tension T equals the resultant compression C; hence the two stresses constitute a couple—"the resisting couple," whose moment is the moment of resistance of the beam at the section in question. These principles will enable us to locate the neutral axis and to determine the resisting moment corresponding to any given

stress on extreme fibre. A convenient method of doing this will be illustrated by an example.

Fig. 6 is a stress-strain diagram of a gravel concrete for both tension and compression. For any section of a beam made of this concrete, the stress diagram is a certain part of the stress-strain diagram, the exact part depending on the loading. Suppose that the loads produce in the lower fibre at the section in question a unit stress represented by bb' say, then T is represented by Nbb' and C by an area Naa' determined from the principle that it must equal the area Nbb'.

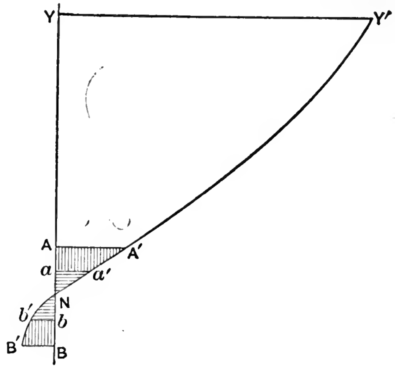


FIG. 6.

Hence the stress diagram is $aa'Nb'b$, and the unit stress on the upper fibre is represented by aa'. Furthermore, ab represents the depth of the beam, and N the position of the neutral axis. Likewise, when the unit stress on the lower fibre is BB' (the ultimate tensile strength) and the beam is on the point of failing, T is represented by the area NBB', and C by the equal area NAA'; hence the stress diagram for the failure stage is $AA'NB'B$, and the unit stress on the upper fibre is AA'.

The resisting moment at the given section is equal to the product of the tension (or compression) and the distance between the centroids of these stresses. For example, at the failure stage of the beam it is equal to the stress area NBB' (or NAA') multiplied by the distance between centers of gravity of these areas. The average ordinate of the area NBB', or average tensile stress, scales 128 lbs/in², and $\overline{NB}=0.6\overline{AB}=0.6d$, where d=depth of beam. Hence if b=breadth of section,

$$C=T=128\times 0.6d\times b=76.8bd.$$

The vertical distance between the centroids of the shaded parts (NAA' and NBB') of the diagram is $0.64\overline{AB}$; hence the arm of the resisting couple is $0.64d$, and the computed ultimate resisting moment of a beam made of the concrete under consideration is $76.8bd \times 0.64d = 49.2bd^2$ in-lbs, b and d to be expressed in inches.

Partly to test the correctness of the theory of flexure of concrete beams, Professor Mörsch made three beams 15×20 cm. in section and several tension and compression specimens of the same mix of concrete. From tests on the specimens he obtained a stress-strain diagram from which he computed the probable resisting moment of the beams to be $3.45bd^2 = 3.45 \times 15 \times 20^2 = 20{,}700$ kg-cm. The average of the actual resisting moments of the beams (determined from tests to destruction) was 22,100 kg-cm., an agreement to be regarded as highly satisfactory.

The working resisting moment of a rectangular beam can be computed from the stress-strain diagram for the material in this same manner. Fortunately, engineers are not called upon to compute resisting moments by this method. It is here set forth principally as a means of introducing important ideas bearing on reinforced-concrete beams.

50. *Inefficiency of Plain Concrete Beams.*—When a beam of the concrete above referred to is loaded to the breaking point, the greatest unit compressive stress in the beam is the stress AA', which is in this case about 375 lbs/in^2. This is very low compared to the ultimate compressive strength (2500 lbs/in^2), and the difference indicates a wasteful use of concrete.

The unshaded portion of the stress-strain diagram (Fig. 6) is also significant in this connection, for it indicates the unused compressive strength of the concrete above the neutral surface when the tensile strength of that below is fully developed and the beam is about to fail.

Another way to express the inefficiency of a concrete beam is to compare its ultimate resisting moment with that which

it would have if the tensile strength and elastic properties were the same as the compressive. On this supposition the tensile stress-strain diagram would be like the compressive; and for the concrete of Fig. 6, the ultimate C and T are represented by the area NYY', and the arm of the resisting couple by twice the vertical distance of the centroid of the area NYY' above N. Actual measurement of the area and distance gives $C=775bd$ and $arm=0.64d$; hence the ideal ultimate resisting moment is $775bd\times0.64d=496bd^2$ as against $49.2bd^2$, the actual value.

To supply the deficiency in tensile strength of concrete is the main purpose of steel reinforcement. A comparatively small amount of steel (rods or bars whose combined sectional area is from 1 to 2 per cent of the total sectional area of the beam) properly embedded will so strengthen the tensile side of the beam that the great strength of the compressive side can be utilized. The exact amount of steel required in any case depends on the elastic properties of the concrete and steel.

51. Varieties of Flexure Formulas.—In the development of the theory of reinforced-concrete beams many different flexure formulas have been used. Some of these have been of an empirical or irrational nature and have been abandoned as the principles underlying the subject have come to be more clearly understood. There still remains, however, a certain variation among rational formulas, these differences being based on differences in fundamental assumptions. These differences relate to: (1) the law of distribution of the compressive fibre stress in the concrete, and (2) the value of the tensile fibre stress in the concrete.

(1) As already explained in Art. 48, the distribution of the compressive fibre stress can be represented by a portion of the stress-strain diagram for the concrete. As shown in Art. 24, the stress-strain curve for concrete up to and even beyond working stresses is nearly straight, and the most widely used flexure formulas for working conditions are based on the assumption that the stress-strain curve is practically straight

up to working stresses. Formulas of Arts. 54–9 and all other flexure formulas of this book (except those of Arts 61–72) are based on this assumption. When the curvature of the stress-strain curve has been taken into account, it has generally been assumed to be an arc of a parabola, the vertex being taken, by some, at the end of curve (the ultimate strength end) and, by others, beyond that point. The formulas of Arts. 61–72 are based on a parabolic stress-strain curve, the vertex being at the end.

(2) As explained in Art. 41, when a reinforced-concrete beam is being loaded, the concrete adjoining the steel generally begins to fail (cracks) before the stress in the steel reaches a value of 5000 lbs/in², and when the stress reaches working values the cracks will have extended well-nigh to the neutral surface. The amount of tension remaining in the concrete at the section of the crack is comparatively small, and this tension being near the neutral surface, the resisting moment due to it is also small compared to that due to the tension in the steel. In a certain formula for ultimate resisting moment in which this residual tension in the concrete is allowed for, the value of the term expressing the contribution of this tension is less than $\frac{1}{2}$ per cent of the total moment. It is the almost universal practice to neglect this tension entirely in flexure formulas; this practice is followed in this book.

Fig. 7 illustrates diagrammatically the various assumptions which may be used in rational flexure formulas. In Fig. 7 (*a*) both the compressive and the tensile stresses are assumed to vary uniformly, as in the usual theory for rectangular beams. This is applicable only to the case of very low stresses in the concrete and in the steel, and would rarely apply to reinforced concrete. In Fig. 7 (*b*) the compressive stresses vary uniformly and the tensile stresses are taken into account for a distance below the neutral axis justified by the permissible deformation of the concrete. With a working steel stress of 15,000 lbs/in², this would not be farther than one-third the distance to the steel. In Fig. 7 (*c*) the tension is neglected and the compression

assumed to vary uniformly. This is the usual assumption for working loads. In Fig. 7 (*d*) a curve is used to represent the compressive stresses. This is usually taken as a parabola with vertex at the ultimate strength of the concrete. For loads below ultimate, the vertex will be above the beam, as in Fig. 7 (*d*). In Fig. 7 (*e*) the full parabola is shown, which corresponds to the ultimate strength of the concrete. In (*d*) and (*e*) the tensile strength may or may not be considered but for high

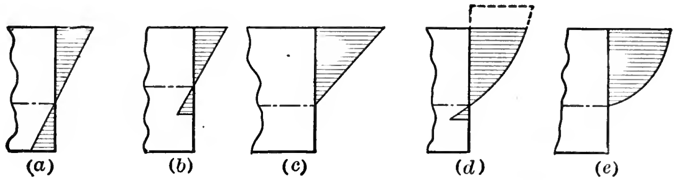


FIG. 7.—Distribution of Fibre Stress in Concrete According to Various Assumptions.

loads this factor becomes of less importance than for working loads and affects the result very slightly.

52. *Use of Various Formulas.*—For use in designing under specified working loads it is the universal practice to use formulas based on the linear law, Fig. 7 (*c*). To calculate ultimate strengths and factors of safety with reference to concrete stresses, formulas based on the full parabola Fig. 7 (*e*), need to be employed; and if the progressive behavior of a beam is to be studied in detail, as in laboratory investigations, it may be desirable to employ formulas based on the more general parabolic law, Fig. 7 (*d*).

53. **Notation.**—Fuller explanations of some of these symbols are given in subsequent articles where the formulas are derived; see also Fig. 8.

f_s = unit fibre stress in steel;
f_c = unit fibre stress in concrete at its compressive face;
e_s = unit elongation of the steel due to f_s;
e_c = unit shortening of the concrete due to f_c;

E_s = modulus of elasticity of the steel;
E_c = modulus of the concrete in compression;
n = ratio E_s/E_c;
T = total tension in steel at a section of the beam;
C = total compression in concrete at a section of the beam;
M_s = resisting moment as determined by steel;
M_c = resisting moment as determined by concrete;
M = bending moment or resisting moment in general;
b = breadth of a rectangular beam;
d = distance from the compressive face to the plane of the steel; = 5, 5.125
k = ratio of the depth of the neutral axis of a section below the top to d;
j = ratio of the arm of the resisting couple to d;
A = area of cross-section of steel;
p = steel ratio, A/bd.

FLEXURE FORMULAS FOR WORKING LOADS.

54. General Assumptions.—The loads being working loads, the unit stress in the steel is within the elastic limit, and the unit stresses in the concrete vary as the ordinates to the compressive stress-strain curve for concrete up to working stresses. This curve is nearly straight; it will be assumed straight to simplify the formulas. The resulting errors are small, as is explained in Art. 74.

Fig. 8 represents a section AB of a beam subjected to bending moment. The neutral axis is at N. The portion of the beam above N is in compression whose maximum value is f_c per unit area, whose average value is $f_c/2$ and whose total value is $\frac{1}{2}f_cbkd = C$. The unit stress in the steel is f_s, and the total steel stress is $f_sA = T$. For the case of simple bending (all loads and reactions transverse to the beam axis), $T = C$, and the resisting moment of the beam is the moment of this couple, $= Tjd = Cjd$. The value of j evidently depends upon

the position of the centroid of the compressive stresses which in turn depends upon the position of the neutral axis. It will be found, further, that the position of the neutral axis depends upon the proportion of steel used and the relative moduli of elasticity of steel and concrete. This position having been found, and from this the value of j, the resisting moment as

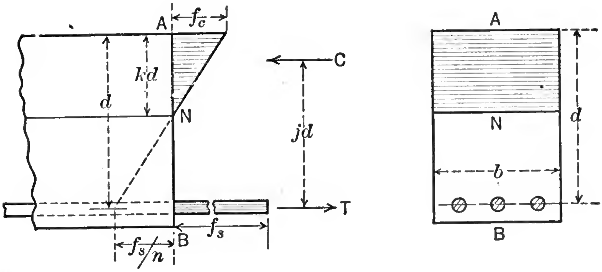


FIG. 8.

shown above is readily calculated for any give values of stress in concrete or steel.

55. Derivation of Formulas.—*Neutral Axis and Arm of Resisting Couple.*—It follows from the assumption of plane sections that the unit deformations of the fibres vary as their distances from the neutral axis; hence, $e_s/e_c = (d-kd)/kd$ (see Fig. 8). Also $e_s = f_s/E_s$ and $e_c = f_c/E_c$; hence, introducing the abbreviation n,

$$\frac{f_s}{nf_c} = \frac{d-kd}{kd} = \frac{1-k}{k}. \quad \text{. . . . .} \quad (a)$$

Then from $T=C$, we have,

$$f_sA = \tfrac{1}{2}f_cbkd. \quad \text{. . . . .} \quad (b)$$

Eliminating f_s/f_c between equations (a) and (b) and introducing the abbreviation p gives $2pn(1-k)=k^2$; this if solved for k gives

$$k = \sqrt{2pn+(pn)^2} - pn. \quad \text{. . . . . .} \quad (1)$$

This formula shows that the position of the neutral axis depends only upon the percentage of steel used and the value of $n=E_s/E_c$, and since E_s is constant (taken at 30,000,000) the value of n depends only upon the character of the concrete.

The distance of the centroid of the compressive stress from the compressive face of the beam is $\frac{1}{3}kd$; therefore the arm of the resisting couple, $T-C$, is given by

$$jd=d-\tfrac{1}{3}kd, \quad \text{or} \quad j=1-\tfrac{1}{3}k. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

In Fig. 9 are given curves for values of k and j for different values of p and n. It should be noted that the value of j does not vary much with p, and that for $n=15$ and p from 0.75 to 1.0%—common values—the average value of j is about 7/8.

56. *Resisting Moment for Given Working Stresses f_s and f_c.*—The safe resisting moment of a given beam may depend upon either the strength of the concrete or the strength of the steel, depending upon the amount of steel actually used. In investigating a given beam we may, therefore, proceed by calculating the safe resisting moment as dependent upon the allowable stress in the concrete and then the resisting moment as dependent upon the allowable stress in the steel and compare results. The smaller of the two values will evidently be the value desired.

The resisting moment in terms of steel stress is

$$M_s=T\cdot jd=f_sA\cdot jd=f_spjbd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In terms of concrete stress the resisting moment is

$$M_c=C\cdot jd=\tfrac{1}{2}f_cbkd\cdot jd=\tfrac{1}{2}f_cjkbd^2. \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

In examining a given design it will generally happen that the values of M_s and M_c will differ, for the reason that the beam was not designed for the precise values of f_s and f_c assumed.

For approximate computations one may use the average

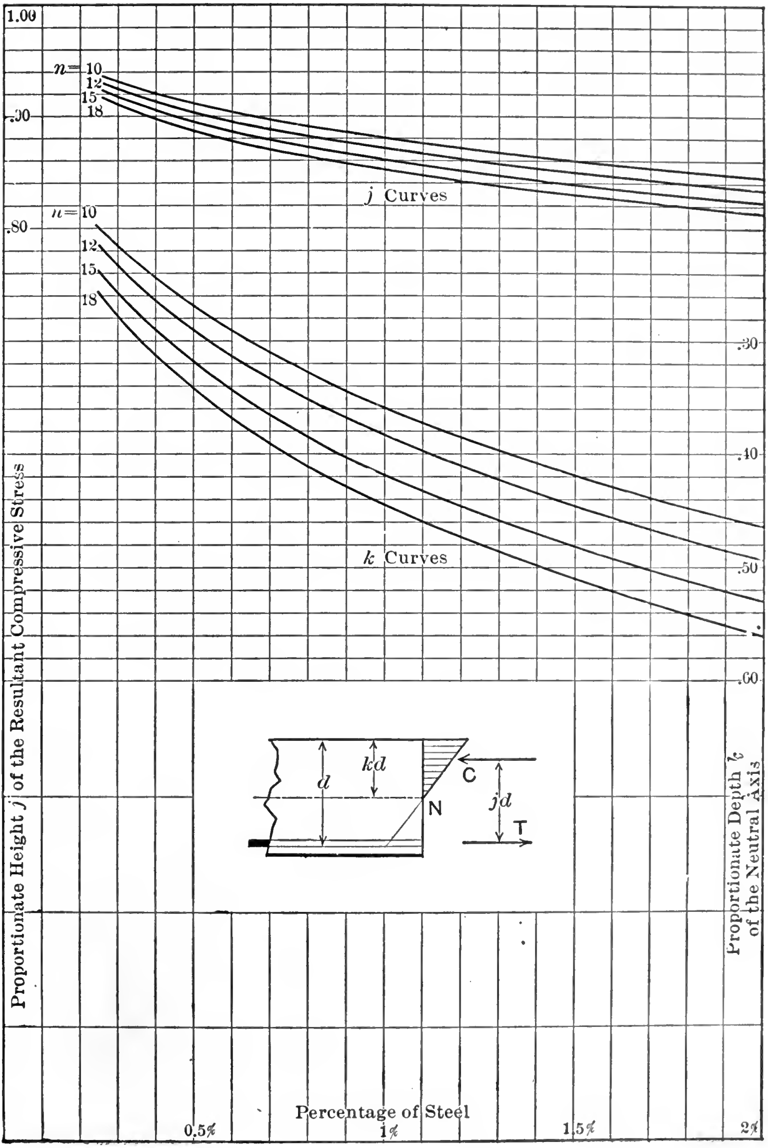

1.00
.90
.80
n = 10
12
15
18
j Curves
n = 10
12
15
18
k Curves
.30
.40
.50
.60
kd
d
C
N
jd
T
Proportionate Height j of the Resultant Compressive Stress
Proportionate Depth k of the Neutral Axis
Percentage of Steel
0.5%
1%
1.5%
2%

FIG. 9.

values $j=\frac{7}{8}$ and $k=\frac{3}{8}$; then formulas (3) and (4) become respectively

$$M_s=f_sA\cdot\tfrac{7}{8}d, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$M_c=f_c\cdot\tfrac{1}{6}bd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

57. *Unit Fibre Stresses for a Given Bending Moment.*—From eqs. (3) and (4) we derive

$$f_s=\frac{M}{Ajd}=\frac{M}{pjbd^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$f_c=\frac{M}{\frac{1}{2}jkbd^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Also the relation between f_s and f_c,

$$f_c=\frac{2f_sp}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Approximating as before, using $j=\frac{7}{8}$ and $k=\frac{3}{8}$, we have

$$f_s=\frac{M}{\frac{7}{8}Ad}; \quad f_c=\frac{M}{\frac{1}{6}bd^2}=\frac{16}{3}f_sp. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

58. *Determination of Amount of Steel and Cross-section of Beam for a Given Bending Moment.*—In designing a beam it is desirable to use the correct amount of steel so that the allowable stresses in both steel and concrete will be reached under the given load; that is, such an amount as will give a *balanced* design. The correct steel ratio can be determined by equating the moments of eqs. (3) and (4), giving $f_sp=\frac{1}{2}f_ck$. Substituting k from eq. (a) and solving for p, we have

$$p=\frac{\frac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c}+1\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This shows that for a given concrete and ratio of working stresses, p has the same value for all sizes of beams. Fig. 10

gives graphically the proper values of p for different ratios f_s/f_c and four different values of n.

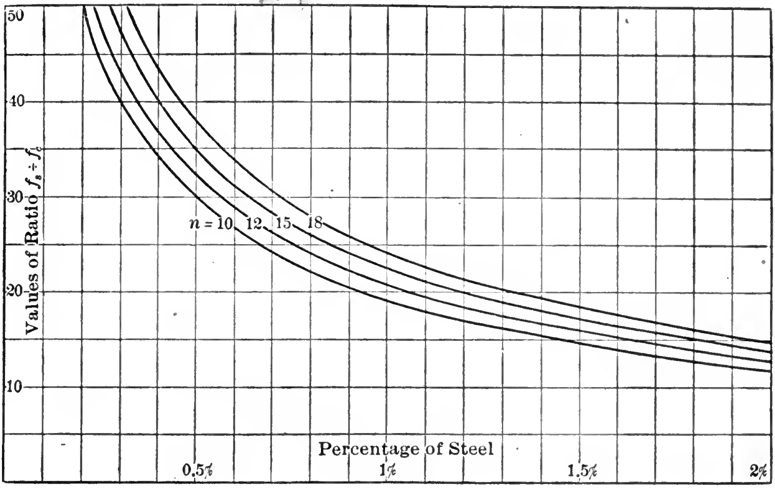


FIG. 10.

If the correct proportion of steel, as given by (11), is used, then the size of the beam, or the value of bd^2, can be determined by either of eqs. (3) or (4). These give

$$\left.\begin{aligned} bd^2 &= \frac{M}{f_s pj} \\ bd^2 &= \frac{M}{\frac{1}{2} f_c kj} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the value of p is taken less than given by (11) then use the first of eqs. (12); if taken greater, then use the second. Equations (12) are convenient to use only when diagrams or tables are available for obtaining the values of p, k, and j. For direct calculation the value of k and j obtained from eqs. (a) and (2) may be substituted in (8), giving

$$bd^2 = \frac{6M\left(\frac{f_s}{f_c} + n\right)^2}{nf_c\left(\frac{3f_s}{f_c} + 2n\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This gives the required dimensions of the beam in terms of the allowable stresses and the value of n, and assumes an amount of steel used as given in eq. (11).

It is to be noted that the application of eqs. (12) or (13) gives the value of bd^2 only, as is the case in the design of any rectangular beam. Having determined bd^2, the values of b and d are to be selected so as to give convenient and economical proportions. The greater the value of d the smaller is the cross-section required and the less the amount of concrete and steel, but the ratio of depth to breadth is limited by practical considerations, such as head room, convenience in placing the steel, and, as shown later, by the shearing-stresses involved. Ordinarily the ratio of $d : b$ will range from about $1\frac{1}{2}$ to 4, the latter for very large beams only. In the case of continuous slabs or floors the width b is, of course, fixed, and the depth only is to be determined.

For Approximate Design assume $k=\frac{3}{8}$ and $j=\frac{7}{8}$. Then eq. (12) gives very closely

$$bd^2=\frac{M}{\frac{1}{6}f_c}=\frac{6M}{f_c}. \qquad (14)$$

which is identical with the usual equation for homogeneous rectangular beams. Having determined bd^2 and the desired dimensions, the stress in the steel and the required area are best found from

$$T=\frac{M}{\frac{7}{8}d} \quad \text{and} \quad A=\frac{T}{f_s}. \qquad (15)$$

59. Examples.—Problems of practical analysis and design are of three general kinds, namely: (1) Given a beam and the safe working stresses, to find the resisting moment of the beam; (2) Given a beam subjected to a given bending moment, to find the unit stresses in concrete and steel; (3) Given the safe working stresses and the bending moment, to find the dimensions of the beam.

(1) A concrete beam is $10'' \times 16''$ in cross-section and the tension reinforcement consists of four $\frac{3}{4}$-inch steel rods, their centres being two inches above the lower face of the beam. The working stress of the con-

crete being 600 lbs/in² and that of the steel 15,000, what is the safe resisting moment of the beam?

Solutions. The cross-section of one steel rod is 0.442 in², hence $A=1.768$; and as $b=10$ and $d=14$, $p=1.768/140=0.0126$. Therefore, n being taken as 15, from (1) $k=0.453$; also from (2) $j=0.849$. As determined by the steel, the resisting moment is (eq. 3)

$$M_s=15{,}000\times1.768\times0.849\times14=315{,}000 \text{ in-lbs.}$$

As determined by the concrete, the resisting moment is (eq. 4)

$$M_c=300\times10\times0.453\times14\times0.849\times14=227{,}000 \text{ in-lbs.}$$

The safe resisting moment is the latter value.

The approximate formulas, (5) and (6), give respectively

$$M_s=15{,}000\times1.768\times\tfrac{7}{8}\times14=325{,}000$$

and $$M_c=600\times\tfrac{1}{6}\times10\times14^2=196{,}000 \text{ in-lbs.}$$

The approximate formula relating to the steel always gives a closer result than the other.

(2) Suppose that the beam of the preceding example is 19 in. deep and is subjected to a bending moment of 350,000 in-lbs. Compute the greatest unit stresses in the steel and concrete.

Solutions. The steel ratio is $1.768/170=0.0104$; and with $n=15$, eq. (1) gives $k=0.424$, and eq. (2) gives $j=0.859$. Therefore $T=350{,}000/0.859\times17=24{,}000$ lbs, and $f_s=24{,}000/1.768=13{,}600$ lbs/in². Also see eq. (8), $f_c=48{,}000/0.424\times10\times17=665$ lbs/in².

The approximate formula (10) gives respectively

$$f_s=13{,}500 \quad\text{and}\quad f_c=750 \text{ lbs/in}^2.$$

Again, of the approximate formulas, the one relating to the steel gives the closer result.

(3) A beam is to be figured to withstand a bending moment of 135,000 in-lbs, the working strength of the concrete and steel being taken at 700 and 12,000 lbs/in² respectively.

Solutions. For $n=15$, eq. (11) gives $p=0.0136$. With this value of p, eq. (1) gives $k=0.462$, and hence $j=0.846$. Eq. (12) now gives

$$bd^2=\frac{135{,}000}{12{,}000\times0.0136\times0.846}=978.$$

Or we may use eq. (13) getting

$$bd^2=\frac{6\times135{,}000(17.1+15)^2}{15\times700(3\times17.1+30)}=978.$$

Many different values of b and d will give the desired value. If b is taken as 7 in., then

$$d^2 = 978/7 = 140, \quad \text{or} \quad d = 12 \text{ in.}$$

Finally $$A = 0.0136(7 \times 12) = 1.14 \text{ in}^2.$$

The approximate formula (10) gives for a suitable steel ratio $p = \frac{3}{16}$ $700/12{,}000 = 0.0109$. Adopting 0.011, then (14) gives $bd^2 = 6 \times 135{,}000/700 = 1157$. Taking $b = 7$ in. as before, $d^2 = 1157/7 = 165.3$, or $d = 12.8$, 13 in. say. Finally $A = 0.011 \times 7 \times 13 = 1.00$ in^2.

60. Diagram for Use in Design.—From eqs. (3) and (4), Art. 56, it is to be seen that

$$\begin{cases} M_s = f_s pj \times bd^2 \\ M_c = \frac{1}{2} f_c kj \times bd^2, \end{cases}$$

in which the quantities $f_s pj$ and $\frac{1}{2} f_c kj$ are variables dependent upon the unit stress, percentage of steel, and value of n; they are independent of the size of beam. For convenience these products will be called *coefficients of resistance* with respect to the steel and the concrete respectively, and will be denoted by R_s and R_c, that is,

$$R_s = f_s pj; \quad R_c = \tfrac{1}{2} f_c kj.$$

Then, for given working stresses,

$$M_s = R_s \cdot bd^2 \quad \text{and} \quad M_c = R_c \cdot bd^2 \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, for any given beam subjected to a bending moment M,

$$R_s = R_c = M/bd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For a balanced design R_s must be equal to R_c and

$$bd^2 = M/R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $R = R_s = R_c$.

Since the value of R for any particular value of n is dependent only upon p and f_s or f_c, it is possible to prepare a convenient diagram from which the values of R can be found. Plates

I and II, drawn for $n=12$ and $n=15$, respectively, are such diagrams. Abscissas represent values of p and ordinates represent values of R_s or R_c for different values of f_s and f_c. In the upper parts of the diagram are given, for convenience, values of k and j, the same as in Fig. 9. Values of R_s are read from the f_s curves and R_c from the f_c curves. For a balanced design $R_s=R_c$ and the correct percentage of steel is found at the intersection of the proper f_s and f_c curves. Thus, in Plate II, for $f_s=16{,}000$ and $f_c=600$, the intersection is a point whose abscissa is 0.68% and ordinate is 94, hence for these stresses the correct percentage of steel is 0.68% and the value of $R_s=R_c=94$. $M=94bd^2$. On the other hand, if the actual percentage in a given beam is, say, 0 8%, and the allowable stresses are $f_s=16{,}000$ and $f_c=700$, then $R_s=110$ and $R_c=117$. This shows at once that the moment of resistance is determined by the steel and the allowable bending moment is $M_s=110bd^2$. To find the actual stresses produced by a given bending moment M, we have $R_s=R_c=M/bd^2$. Calculating this value from the given data we enter the diagram with this value of R and the given value of p and find the corresponding values of f_s and f_c.*

Examples.—The examples of Art. 59 will now be solved by the use of the diagram, Plate II.

(1) The percentage of steel being 1.26, we first find that value on the lower margin; then trace vertically, stopping at the first of the two curves $f_c=600$ and $f_s=15{,}000$; then trace horizontally to either side margin and read off the value $R=115$. Finally $M=115\times10\times14^2=225{,}400$ in-lbs.

(2) $R=M/bd^2=350{,}000/10\cdot17^2=121$, and the percentage of steel is 1.04. We enter the diagram with these values of R and p, find the intersection of the horizontal and vertical lines through these values respectively, and from the steel and concrete curves adjacent to this intersection estimate f_s to be 13,750 and f_c 675 lbs/in².

(3) We first find the intersection of the curves $f_c=700$ and $f_s=12{,}000$; from that point tracing down we find $p=1.35\%$, and tracing horizontally we find $R=137$. Then $bd^2=M/R=135{,}000/137=986$, from which b and d may be decided upon, and then finally the amount of steel.

* These diagrams are modeled after those contributed by Prof. French in Trans. Am. Soc. C. E., Vol. LVI, 1906, p. 362.

FLEXURE FORMULAS FOR ULTIMATE LOADS.

61. General Assumptions.—It is assumed that the amount of reinforcement is sufficient to develop the full compressive strength of the concrete without straining the steel beyond its yield point; or otherwise expressed, failure occurs by crushing of the concrete, the stress in the steel being still within the yield point. The variation of compressive stress will then be represented in Fig. 11 by a full parabola with vertex at E and horizontal axis AD.

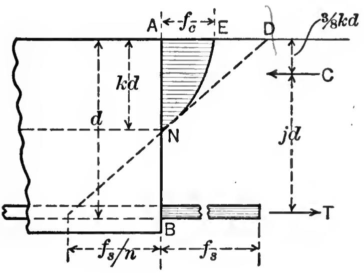


FIG. 11.

In the present connection, the two following properties of a parabola like that of Fig. 11 are useful: (1) The average abscissa of the parabolic arc equals two-thirds the greatest, f_c; (2) the distance from the centroid of the parabolic area to its top equals three-eighths the total height, kd.

62 Derivation of Formulas. — *Neutral Axis and Arm of Resisting Couple.*—The "initial modulus of elasticity" of the concrete (Art. 25) is denoted by E_c in the present article. It is represented by the slope of the tangent ND with respect to the deformation axis NA, or is equal to AD/NA. For a parabola, vertex at E, $AD=2AE$, and since $e_c=NA$, we have $f_c=\frac{1}{2}E_ce_c$. Also, $f_s=E_se_s$, and from the assumption of plane sections it follows that $e_s/e_c=(d-kd)/kd$. Eliminating e_s/e_c from the above equations, and introducing the abbreviation n, gives

$$\frac{f_s}{2nf_c}=\frac{1-k}{k}. \qquad \qquad (a)$$

Then, from $T=C$,

$$f_spbd=\tfrac{2}{3}f_cbkd. \qquad \qquad (b)$$

Eliminating f_s/f_c between eqs. (a) and (b) and introducing

the abbreviation p, gives $3pn = k^2/(1-k)$; this, if solved for k, gives

$$k = \sqrt{3pn + (\tfrac{3}{2}pn)^2} - \tfrac{3}{2}pn. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The distance of the centroid of the compressive stress from the compressive face of the beam is $\tfrac{3}{8}kd$; therefore, the arm of the resisting couple $T-C$ is given by

$$jd = d - \tfrac{3}{8}kd, \quad \text{or} \quad j = 1 - \tfrac{3}{8}k. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Fig. 12 gives curves for values of k and j for various values of p and n.

Comparing these curves with those of Fig. 9, it will be noted that the values of k and j are somewhat less, showing that the neutral axis is a little lower under the parabolic assumption of stress variation than under the straight line assumption.

63. *Ultimate Resisting Moment for a Given Ultimate Strength* f_c.—Remembering the assumption made at the outset in regard to the amount of steel (Art. 61), it will be understood that the ultimate resisting moment always depends on the concrete; the value is

$$M_c = C \cdot jd = \tfrac{2}{3} f_c bkd \cdot jd = \tfrac{2}{3} jkf_c bd^2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

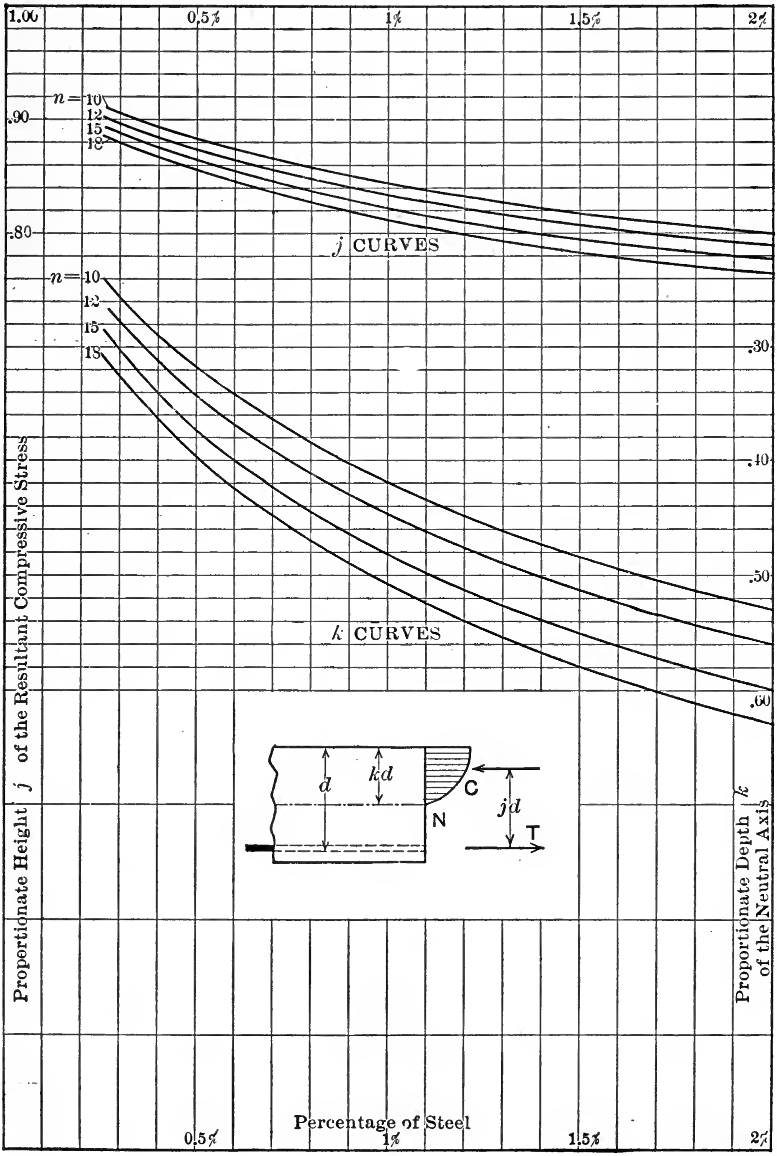
It should be remembered that this equation gives the ultimate resisting moment only if when the unit stress in the concrete is at the ultimate that in the steel is not beyond the elastic limit.

If f_s is the steel stress corresponding to f_c we have

$$M_s = Tjd = f_s pjbd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Placing $M_s = M_c$ of eq. (3) the value of f_s can be found.

If this calculated value of f_s is less than the elastic limit of the steel, then the ultimate resisting moment is correctly given by (3); but if it exceeds the elastic limit, then the beam fails by over-stressing of the steel and the formulas of this article do not strictly apply. (See Arts. 68–72 for a solution


1.00
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.60
0.5%
1%
1.5%
2%
$n=10$
12
15
18
j CURVES
k CURVES
d
kd
C
N
jd
T
Proportionate Height j of the Resultant Compressive Stress
Proportionate Depth k of the Neutral Axis
Percentage of Steel

FIG. 12.

of this case.) However, as the steel stress in eq. (4) depends only upon the value of j, and as this does not vary greatly for the extreme cases of working loads (Fig. 9), and ultimate loads (Fig. 12), the steel stress will be closely given by eq. (4) for all cases.

64. *Determination of Amount of Steel and Cross-section of Beam for a Given Ultimate Bending Moment.*—To find the proper amount of steel to correspond to given values of f_c and f_s, place M_s of eq. (4) equal to M_c of eq. (3) and substitute the value of k as given by equation (a). Solving for p, there results

$$p=\frac{2/3}{\frac{f_s}{f_c}\left(\frac{f_s}{2nf_c}+1\right)} \qquad (5)$$

Fig. 13 gives graphically the value of p for various values of the ratio f_s/f_c and four different values of n. It shows what

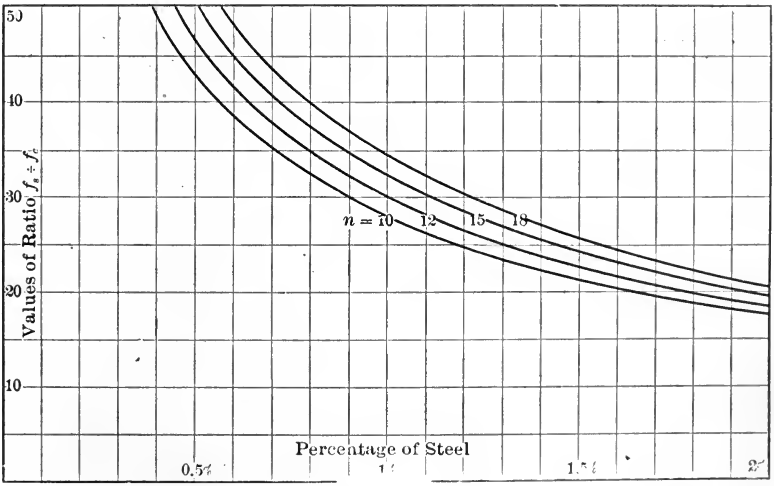


FIG. 13.

amount of steel would be required to cause the ultimate strength of the concrete to be reached simultaneously with any given stress f_s in the steel, not exceeding the elastic limit.

If, in any given case, the steel ratio as given by (5), or a higher value, is adopted, then the concrete would crush without straining the steel beyond the elastic limit, and the ultimate resisting moment of the beam is given by (3), which value equated to the ultimate bending moment, M, to be provided for, gives $\frac{2}{3}f_cjkbd^2=M$, or

$$bd^2=\frac{M}{\frac{2}{3}f_cjk}. \qquad \qquad (6)$$

From this, d may be computed for any assumed value of b.

If the percentage of steel used is less than that given by (5), with f_s = elastic limit, then the ultimate resisting moment can be only approximately calculated by eq. (4), as indicated in Art. 63.

65. Diagrams for Use in Calculations.—Assuming the steel stress to remain always within the elastic limit, the factors f_spj and $\frac{2}{3}f_cjk$ of eqs. (4) and (3) of Art. 63 may be called "coefficients of resistance," as in Art. 60. Or

$$R_s=f_spj; \quad R_c=\tfrac{2}{3}f_ckj. \qquad \qquad (1)$$

The diagram of Plate III gives values of R_s and R_c for various values of f_s, f_c, and p. The f_c curves are given for $n=15$ and $n=12$; the f_s curves are given for $n=15$ only, but the values for $n=12$ differ very little from those for $n=15$, so that the diagram may be used for both values of n. In this case it is to be noted that the modulus of elasticity assumed in the equations is the initial modulus, or tangent modulus at the origin. This is in general somewhat higher than the secant modulus suitable for use in working formulas. The initial modulus may generally be taken at about 2,500,000, giving a value of $n=12$.

66. Examples.—As before, problems group themselves into three classes, namely:

(1) Given a beam and the ultimate stresses, to find the ultimate resisting moment of the beam.

(2) Given a beam, and its ultimate bending moment, to find the unit stresses in the concrete and steel.

(3) Given the ultimate stresses, the factor of safety and the bending moment to be carried, to find the dimensions of the beam.

(1*a*) A concrete beam is 10×16 inches in cross-section and the tension reinforcement consists of four $\frac{7}{8}$-inch steel rods, their centers being 2 inches above the lower face of the beam. The ultimate compressive strength of the concrete being 2000 and the elastic limit of the steel 45,000 lbs/in², compute the ultimate resisting moment of the beam. Assume $n=15$.

Solution. Here $p=0.0171$, and for $n=15$, eq. (1) gives $k=0.570$ and (2) gives $j=0.785$. Hence

$$M_c = \tfrac{2}{3}2000 \times 0.570 \times 0.785 \times 10 \times 14^2 = 1{,}170{,}000 \text{ in-lbs.}$$

It remains to test whether the stress in the steel would be within the elastic limit, the beam being subjected to a bending moment of 1,170,000 in-lbs. This is done by dividing the bending moment by the arm of the resisting couple, which gives the whole tension in the steel, and then this tension by the area of the steel; thus

$$\frac{1{,}170{,}000}{0.785 \times 14} = 106{,}500 \text{ lbs} = T,$$

$$\frac{106{,}500}{2.404} = 44{,}200 \text{ lbs/in}^2 = f_s.$$

This result being below the elastic limit, the ultimate resisting moment depends upon the concrete and is correctly given above.

(1*b*) Suppose the beam of problem (1) to be reinforced with four $\frac{3}{4}$-inch steel rods of elastic limit equal to 40,000 lbs/in². All other things remaining the same, compute the ultimate resisting moment of the beam.

Solution. $p=0.0126$ and for $n=15$, $k=0.52$ and $j=0.805$

$$M_c = \tfrac{2}{3} \times 2000 \times 0.52 \times 0.805 \times 10 \times 14^2 = 1{,}096{,}000 \text{ in-lbs.}$$

The corresponding stress in the steel from eq. (4) is

$$f_s = \frac{1{,}096{,}000}{1.768 \times 0.805 \times 14} = 55{,}000 \text{ lbs/in}^2.$$

This result is beyond the stated elastic limit, and eq. (3) does not apply to the problem in hand. The ultimate resisting moment can be found approximately by using value of $j=0.805$ above given. Then with $f_s=40{,}000$, we have

$$M_s = 40{,}000 \times 1.768 \times 0.805 \times 14 = 800{,}000 \text{ in-lbs.}$$

See example (2) Art. 73 for a more exact solution.

(2) A beam 12×22 inches in cross-section reinforced with four 1-inch round rods placed 2 inches above the bottom of the beam, fails by crush-

ing of the concrete at an ultimate bending moment of 2,700,000 in-lbs. Find the unit stresses in the concrete and steel at failure. Assume $n=12$.

Solution. Here $p=0.0131$, and for $n=12$, eq. (1) gives $k=0.490$ and eq. (2) gives $j=0.815$.

From eq. (3)

$$f_c=\frac{M}{\frac{2}{3}kjbd^2}=\frac{\frac{3}{2}\cdot 2{,}700{,}000}{0.490\times 0.815\times 12\times 20^2}=2140 \text{ lbs/in}^2$$

and from eq. (4)

$$f_s=\frac{M}{Ajd}=\frac{2{,}700{,}000}{3.14\times 0.815\times 20}=52{,}700 \text{ lbs/in}^2.$$

(3) Design a beam to carry a safe bending moment of 600,000 in-lbs. with a factor of safety of 3. The ultimate strength of the concrete is 2100 lbs/in², the elastic limit of the steel is 48,000 lbs/in², and $n=12$.

Solution. $f_s/f_c=22.9$. Eq. (5) gives $p=0.0149$ as the correct steel ratio.

Eq. (1) gives $k=0.515$ and (2) gives $j=0.805$.

Eq. (6) $$bd^2=\frac{M}{\frac{2}{3}f_ckj}=\frac{3\times 600{,}000}{\frac{2}{3}\times 2100\times 0.515\times 0.805}=3100.$$

Trying 18 inches for d, then $b=9.55=9\frac{1}{2}$ inches. Also $A=.0149\times 18\times 9.55=2.55$ in².

The preceding examples will now be solved by the use of diagrams.

(1*a*) The percentage of steel being 1.71, we first find that value on the lower margin of the diagram, and then trace vertically to the solid line marked $f_c=2000$. We note that the point thus found is below the line $f_s=45{,}000$, the elastic limit of the steel, and hence conclude that the amount of steel in this beam is sufficient to develop the full compressive strength of the concrete. We than trace horizontally from the point as found above to either side of the diagram and read $R=600$. Then $M=Rbd^2=600\times 10\times 14^2=1{,}175{,}000$ in-lbs.

(1*b*) Proceeding as in problem (1*a*) we find that the point on the diagram determined by $p=1.26\%$, $f_c=2000$ and $n=15$ corresponds to a value of $f_s=55{,}000$ lbs/in². Since this value is above the elastic limit of the steel in the beam we conclude that the ultimate resisting moment depends on the steel and can be only approximately determined from this diagram. For $p=1.26\%$ and $f_s=40{,}000$ we find $R=410$ and hence the resisting moment is approximately equal to $410\times 10\times 14^2=803{,}000$ in-lbs.

(2) Having given the dimensions of the beam and the ultimate resisting moment we determine R to be 562 from $R=M\div bd^2$, and $p=1.31\%$.

The point on the diagram fixed by $R=562$ and $p=1.31\%$ corresponds to a steel stress of 53,000 lbs/in² and a concrete stress of 2100 lbs/in², assuming $n=12$.

(3) We first find the intersection of the curves $f_c=2100$ and $f_s=48{,}000$ interpolating between the curves drawn, assuming $n=12$; from that point tracing down we find $p=1.5\%$ and horizontally we find $R=575$. Then $bd^2=M/R=1{,}800{,}000/575=3130$, from which d and b can be decided upon, and finally the amount of steel.

67. Comparison of Ultimate Strength and Safe Strength. Factor of Safety.—When a beam is progressively loaded, the stresses in the concrete and steel will increase in proportion to the load and bending moment until the stresses in the concrete have become so great that the law of stress variation therein begins to depart sensibly from the straight-line law. Beyond this point the stress in the concrete will increase at a lower rate than the load and the departure from the law of proportionality will become greater as the load increases. When the maximum fibre stress in the concrete has reached its ultimate strength, then, according to the parabolic law, the maximum stress is only $1\frac{1}{2}$ times the average instead of twice the average, as in the case of the straight-line law. The stress in the steel is very nearly proportional to the load, as the only factor causing a variation from this law is the slight change in value of j.

From these considerations it appears that the factor of safety against ultimate failure from crushing of the concrete is considerably more than is indicated by the ratio of ultimate crushing strength of the concrete to the compressive stress used as the working stress. Thus, if the ultimate strength of a concrete is 2000 lbs/in², and 500 lbs/in² is used for the working stress, the actual factor of safety of the beam against failure by crushing of the concrete will be much more than 2000/500 or four. It will be nearer *six*. To illustrate this condition, we will calculate the factor of safety of the beam in Ex. (3), Art. 59, with respect to crushing of the concrete. Assuming the ultimate strength of the concrete to be 2000 lbs/in².

Solution. The bending moment is 135,000 in-lbs. and $bd^2=7\times12^2=1008$, $p=1.36$. Then $M/bd^2=134$. From Plate III the value of R for $p=1.36$ and $f_c=2000$ $(n=12)$ is about 540. Then the factor of safety is $540/134=4.03$; that is the breaking load is 4.03 times the safe load. Comparing the working stress of 700 with the ultimate of 2000 the ratio is only 2.86, thus the actual factor of safety is 1.41 times the ratio of stresses. Note from the diagram, Plate III, that the corresponding stress in the steel is 50,000 lbs/in² which is 4.2 times the working stress. Thus the steel stress varies nearly in the same proportion as the moment.

GENERAL FLEXURE FORMULAS FOR ANY LOAD.

68. General Assumptions.—The two sets of formulas already derived are sufficient for all purposes of design and for estimates of ultimate strength. For the purpose of a more scientific study of various problems arising in the laboratory or elsewhere, it will be convenient to have available formulas based on the more general parabolic curve assumed in Fig. 7*d*. In what follows the form of treatment follows that developed by Professor Talbot. It is assumed that the stress in the steel is not above the yield point. The parabola representing the variation of compressive stress is not a "full one," that is, its top is not the vertex, see Fig. 14, unless the maximum concrete stress is at the ultimate value. As heretofore, f_c and e_c will denote the unit stress and strain respectively at

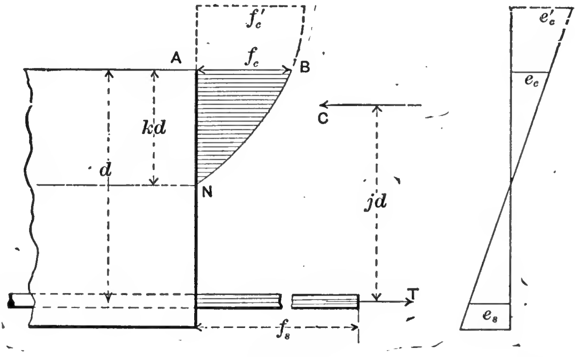


Fig. 14.

the compressive face of the concrete, and as in Art. 62, E_c will denote the initial modulus of elasticity of the concrete. In this article f_c' and e_c' will denote these same quantities at

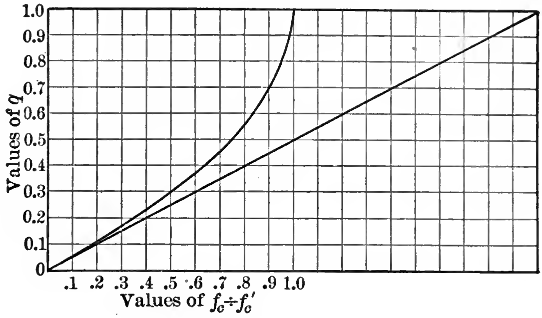


FIG. 15*a*.

the ultimate stage of the concrete, and q will be used as an abbreviation for e_c/e_c'. It can be shown from the properties

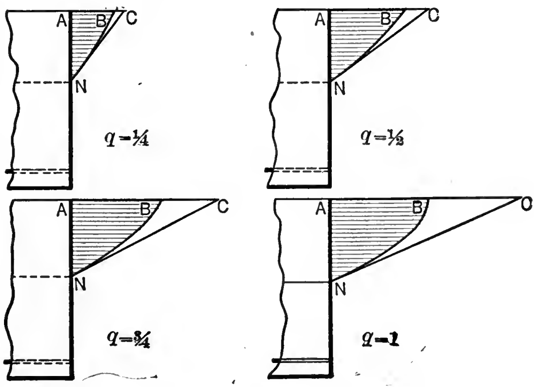


FIG. 15*b*.

of a parabola that: (1) The average abscissa to the parabola NB is $(3-q)/3(2-q)$ times the greatest abscissa f_c; (2) the distance from the centroid of the parabolic area to the top AB is $(4-q)/4(3-q)$ times its height, kd; and (3)

$$f_c = f_c'(2-q)q = \tfrac{1}{2}(2-q)E_c e_c. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Fig. 15*a* shows graphically the relation between q and the ratio f_c/f_c'; thus, when $q=\frac{1}{4}$ (the concrete is strained to one-fourth its limit of compression) the unit stress in the concrete is about 0.45 of the ultimate strength.

The lines NB in Fig. 15*b* show the distributions of compressive stress at a section of a beam when q is $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 respectively as marked. In each case N is the neutral axis and AB represents the unit stress on the remotest fibre. When q is $\frac{1}{4}$, the distribution is almost linear.

69. Derivation of Formulas.—*Neutral Axis and Arm of Resisting Couple.*—As in Arts. 55 and 62, $e_s/e_c=(d-kd)/kd$, and $f_s=E_se_s$. Eliminating e_s/e_c from these two equations and (a), and introducing the abbreviation n, gives

$$\frac{f_s}{nf_c}=\frac{2(1-k)}{k(2-q)}. \quad \cdots\cdots \quad (b)$$

Equating total tension and total compression,

$$Af_s=kbdf_c(3-q)/3(2-q). \quad \cdots\cdots \quad (c)$$

Eliminating the ratio f_s/f_c between equations (b) and (c), and introducing the abbreviation p, gives $6pn(1-k)=k^2(3-q)$, which, solved for k, furnishes the following formula:

$$k=\sqrt{2\frac{3pn}{3-q}+\left(\frac{3pn}{3-q}\right)^2}-\frac{3pn}{3-q}. \quad \cdots\cdots \quad (1)$$

The distance of the centroid of the compressive stress from the top of the beam is $kd(4-q)/4(3-q)$; hence the arm of the resisting couple is given by $jd=d-kd(4-q)/4(3-q)$ or

$$j=1-\frac{k(4-q)}{4(3-q)}. \quad \cdots\cdots \quad (2)$$

Fig. 16 gives values of k and j for various values of p and for $q=\frac{1}{4}$; and Fig. 17 shows how k and j varies with q, for $n=15$.

70. *Resisting Moment for Given Values of f_c and f_s.*—Whether the resisting moment is determined by the concrete or steel

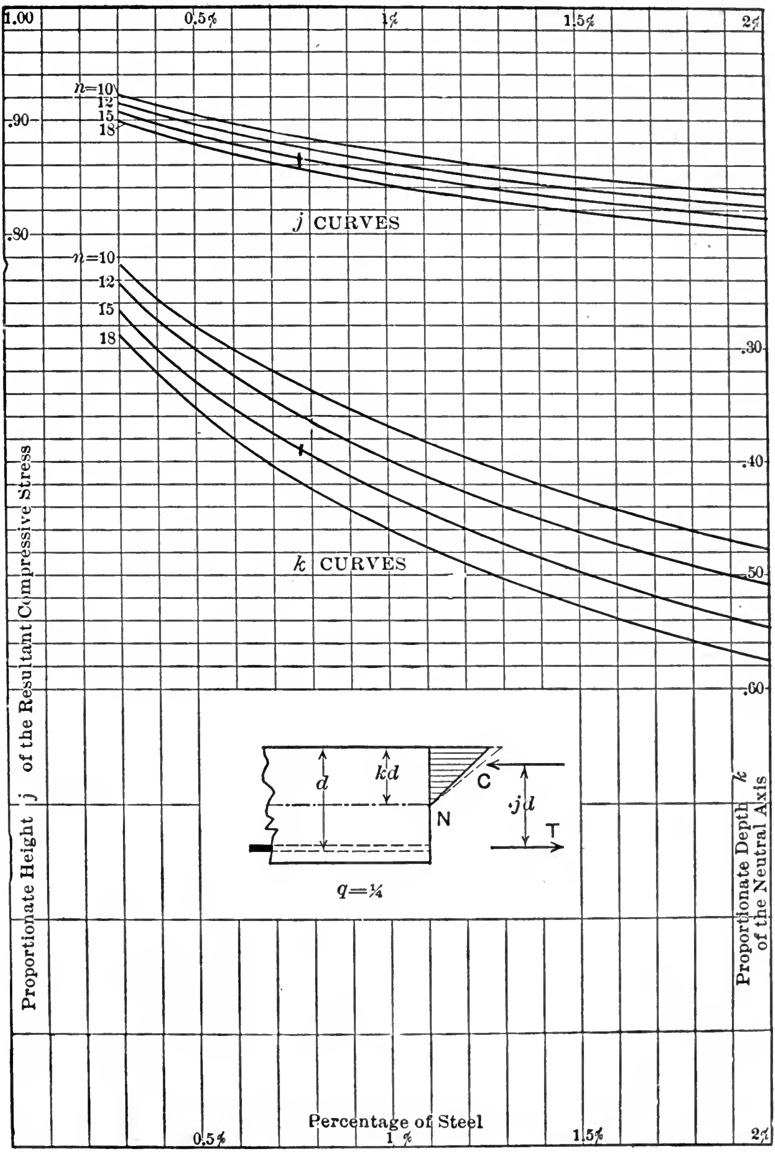

1.00
0.5%
1%
1.5%
2%
.90
.80
n=10
12
15
18
j CURVES
n=10
12
15
18
.30
.40
k CURVES
.50
.60
Proportionate Height j of the Resultant Compressive Stress
Proportionate Depth k of the Neutral Axis
d
kd
C
N
jd
T
q=¼
Percentage of Steel
0.5%
1 %
1.5%
2%

Fig. 16.

depends on the percentage of reinforcement; in a general way the higher percentages make the moment depend on the concrete and the lower on the steel. As depending on the concrete, the resisting moment is given by

$$M_c = Cjd = \frac{3-q}{3(2-q)} jkf_c bd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The value of q to be used here must correspond with the f_c used, the relation between q and f_c being given by (a) of Art. 68, or by Fig. 15a. As depending on the steel, the resisting moment is

$$M_s = Tjd = f_s Ajd = f_s pjbd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

71. *Determination of Fibre Stresses f_s and f_c for a Given Bending Moment.*—Formulas for these can be obtained by solving (3) and (4) for f_c and f_s respectively; thus

$$\left.\begin{aligned} f_c &= \frac{3(2-q)}{(3-q)} \frac{M}{jkbd^2} \\ f_s &= \frac{M}{pjbd^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Neither f_c nor f_s can be determined directly from these, for each formula contains q (j and k depend on q), which is an unknown in the problem in hand. An estimated value of q must be used for a trial solution of (5), and then with the value of f_c thus found a better value of q may be obtained from (a) or from Fig. 15a, which value may be used in a second trial solution.

72. *Determination of Amount of Steel and Cross-section of Beam for a Given Bending Moment.*—In order that the maximum unit compression in the concrete, f_c, and the unit stress in the steel, f_s, may have certain definite values when the beam is subjected to a given bending moment, a certain definite percentage of steel must be used. This percentage is such as makes the values of the resisting moment as determined by steel and concrete equal. Thus equating values of M from equations (3) and (4) and simplifying,

$$p = (3-q)kf_c / 3(2-q)f_s.$$

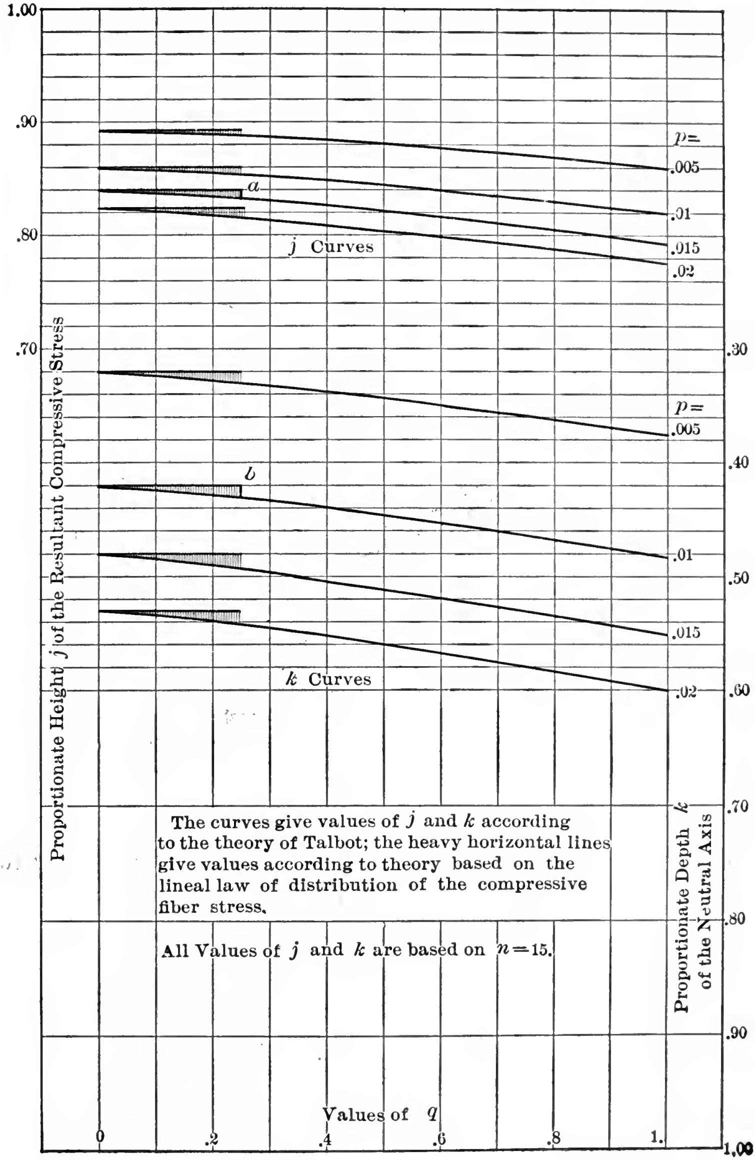

1.00
.90
.80
.70
Proportionate Height j of the Resultant Compressive Stress
p=
.005
.01
.015
.02
a
j Curves
.30
p=
.005
.40
b
.01
.50
.015
k Curves
.02
.60
.70
.80
.90
1.00
Proportionate Depth k of the Neutral Axis
The curves give values of j and k according to the theory of Talbot; the heavy horizontal lines give values according to theory based on the lineal law of distribution of the compressive fiber stress.
All Values of j and k are based on n=15.
Values of q
0
.2
.4
.6
.8
1.

Fig. 17.

Inserting in this the value of k furnished by (b) gives

$$p=\frac{3-q}{3(2-q)}\,\frac{1}{\frac{f_s}{f_c}\left(\frac{2-q}{2n}\frac{f_s}{f_c}+1\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

In this also the value of q used should correspond to the value of f_c adopted as working stress. The curves of Fig. 18 give

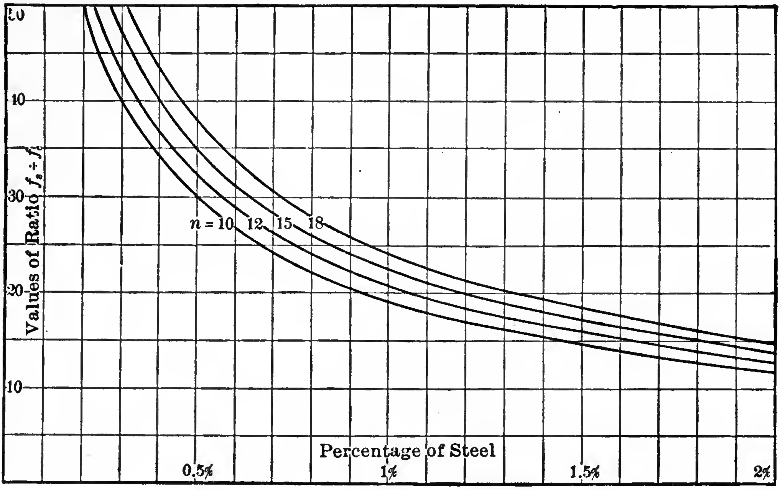


FIG. 18.

values of p for different values of f_s/f_c up to 50, q being taken at $\frac{1}{4}$.

If in any given case a value for p less than that given by (6) is adopted, then the resisting moment is given by equation (4), which equated to the bending moment to be provided for gives $f_s pjbd^2=M$, or

$$bd^2=\frac{M}{f_s pj}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

If a greater value of p is adopted, then the resisting moment

is given by (3), which if equated to the bending moment gives $jkf_cbd^2(3-q)/3(2-q)=M$, or

$$bd^2=\frac{3(2-q)}{3-q}\frac{M}{jkf_c}. \qquad (8)$$

From the proper one of these, d can be computed for any assumed value of b.

73. **Examples.**—(1) It is required to solve example 1, Art. 59, by the methods of this article, it being supposed that for the working stress $f_c=600$ lbs/in², $q=\frac{1}{4}$.

Solution. As shown in the solution of the example referred to, $A=1.768$ in², and $p=0.0126$; therefore from eq. (1) or Fig. 16, n being taken as 15, $k=0.466$, and from eq. (2) or Fig. 16, $j=0.842$. Then from eqs. (3) and (4)

$$M_c=\tfrac{11}{21}\times 0.842\times 0.466\times 600\times 10\times 14^2=242{,}000 \text{ in-lbs.}$$

and $$M_s=15{,}000\times 1.768\times 0.842\times 14=313{,}000 \text{ in-lbs.}$$

(2) It is required to solve example 2 of Art. 66 by the methods of this article.

Solution. As disclosed by the solution in Art. 66, the stress in the steel will reach the elastic limit before that in the concrete would reach the ultimate strength; hence the ultimate resisting moment depends on the steel. The stress existing in the concrete when the steel is stressed to the elastic limit is unknown; so is q. Supposing that this stress in concrete is $\frac{3}{4}$ the ultimate strength, $q=0.5$ (see Fig. 15a); then, since $p=0.0126$, and n is taken as 15, $k=0.48$ and $j=0.83$ (see Fig. 17), and eq. (4) gives $M_s=820{,}000$ in-lbs. For a bending moment of this value, the stress in the concrete would be (with the above values of q, j, and k) 1260 lbs/in² (see eq. 5). Now for the ratio 1260/2000, q is about 0.4, k, 0.75, and j, 0.825. Since this value of j is practically like the one used in the trial computation, the ultimate resisting moment may be taken as 820,000 in-lbs. Compare with results in Art. 66.

(3) It is required to solve example 2 of Art. 59 by the methods of this article, supposing the ultimate compressive strength of concrete to be 2500 lbs/in².

Solution. This problem can only be solved by trial because it is necessary to know q at the outset, and q depends on a quantity sought, f_c. Supposing that the load is about a safe one, then q equals about $\frac{1}{4}$. With this value, n equal to 15, and p equal to 0.0104 (already found

on page 61), $k=0.43$, and $j=0.85$ (see Fig. 16). Then eq. (5) gives $f_c=630$ lbs/in². Now q depends on the ratio of the working stress in the concrete to its ultimate strength; for the approximate value, 630, the ratio is 0.25, and eq. (a), or Fig. 15a, gives $q=0.15$. With this value eq. (1) gives $k=0.432$, eq. (2), $j=0.854$ (see also Fig. 17), and eq. (5), $f_c=635$ lbs/in². This value is so near the first that $q=0.15$ must be practically correct, and $j=0.854$ may be used to determine the stress in the steel. For this, eq. (5) gives $f_s=13{,}700$ lbs/in².

(4) It is required to solve example 3 of Art. 59 by the methods of this article, the ultimate compressive strength of the concrete being taken at 2000 lbs/in².

Solution. For the ratio 700/2000, q is about 0.2 (see Fig. 15a). With $n=15$ eq. (6) gives $p=0.018$. For this value of p, we may use either (7) or (8) to compute the dimensions of the section. Choosing (7) we need first a value of j, which may be obtained from (2) and (1), or closely enough from Figs. 16 and 17; the figures give $j=0.82$, and eq. (7) gives $bd^2=763$. With $b=7$ (as in Art. 59) d is 10.5 in.

74. Comparison of Flexure Formulas after Talbot with (1) those for working conditions as given in Art. 55–58, and (2) those for ultimate conditions as given in Art. 62–64:

(1) The heavy horizontal lines of Fig. 17 give values of j and k, according to the linear law (Art. 55), and the curved lines those after Talbot. For $q=0.25$ and $p=0.015$, the difference between the two values of j is represented by a and the difference between the two values of k by b. For all values of q up to 0.25 or 0.30 the first difference is small, and so the values given by the two formulas for f_s must be nearly the same. The second difference is larger, and the two formulas for f_c will not agree so closely. An exact comparison will now be made.

Art. 57 gives (see eqs. 7 and 8):

$$f_s'=\frac{M}{pj'bd^2} \quad \text{and} \quad f_c'=\frac{2M}{j'k'bd^2}.$$

(The primes are used to distinguish the symbols from the corresponding ones in the other formulas.) Comparing these with eq. (5), Art. 71, one gets

$$\frac{f_s'}{f_s}=\frac{j}{j'} \quad \text{and} \quad \frac{f_c'}{f_c}=\frac{2(3-q)}{3(2-q)}\frac{jk}{j'k'}.$$

As already explained, q rarely exceeds $\frac{1}{4}$ for working conditions; with this value and $n=15$, the following table gives the ratios f_s'/f_s and f_c'/f_c for five percentages of steel. For values of q less than $\frac{1}{4}$, the ratios are nearer unity; for $q=0$, they are all unity and the two sets of formulas are identical.

$p=$	$\frac{1}{4}$%.	$\frac{1}{2}$%.	1%.	1.5%.	2%.
f_s'/f_s	0.995	0.993	0.991	0.990	0.989
f_c'/f_c	1.092	1.091	1.090	1.088	1.086

The unit stresses in the steel as given by the two formulas are practically identical. Any error involved in the formulas for f_c', based on the linear law, is on the side of safety.

(2) For loads which stress the concrete to the ultimate limit, the stress parabola of Fig. 14 is full like that of Fig. 11, and $q=1$. The formulas of Arts. 68–72 for this stage and those of Arts. 62–64 are identical.

FLEXURE FORMULAS FOR T-BEAMS.

75. Use of T-Beams.—Where a concrete floor slab is constructed integrally with the supporting beams so that unity of action is insured, then the beam, with a portion of the slab above it, constitutes a so-called T-beam (Fig. 19). In regions of positive bending moment the slab acts as the *compression flange*, the steel being placed near the bottom. The narrow part of the beam is commonly called the *web*. In regions of negative bending moment (near the support in the case of continuous beams) the slab is in tension and the lower part of the beam is in compression. If it be assumed that the concrete takes no tension, the beam becomes a rectangular beam with tension side uppermost, and the principal reinforcing steel is placed near the top. Some of the steel is also generally continued straight through near the bottom, thus reinforcing the beam in compression. (See Art. 84.)

The portion of the slab which may be assumed to act in compression, or the effective width of the flange, should be

limited to about five or six times its thickness, but, of course, not exceeding the distance center to center of beams. (See Art. 150 for further discussion of this subject.)

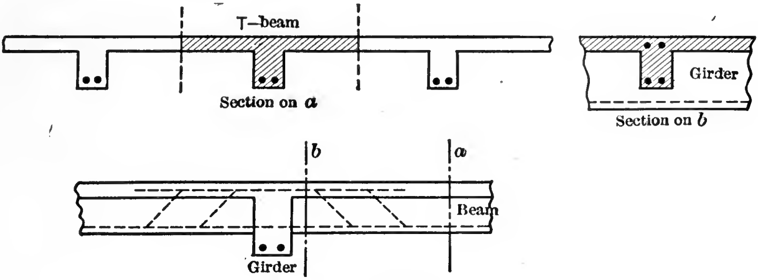


FIG. 19.

T-beams are also used occasionally where not constructed in connection with a floor slab. Since the concrete in the lower part of a beam takes no tension its only duty is to transmit stress from tension steel to compression concrete (involving mainly shearing-stress) and for this purpose the entire rectangular section is not needed in large beams. Economy can thus be secured by omitting a portion of the concrete, leaving a T-form of section.

76. Assumptions and Notation.—The neutral axis of a T-beam may lie in the *flange* or below the flange in the *web*, depending upon the relative depth of flange and beam, and amount of steel used. If the neutral axis is in the flange then the formulas for rectangular beams apply as the concrete below the neutral axis is of no significance. If it lies in the web the compression area is of different form from that in rectangular beams and different formulas are required. These are developed in the following articles.

Whether or not the neutral axis lies in the flange can readily be determined by means of the formulas and diagrams for rectangular beams, especially Plate II. Having given the amount of reinforcement, or the values of f_c and f_s, the value of k is determined and compared with the slab thickness. The

same information can be obtained also by use of the formulas and diagrams for T-beams as explained later.

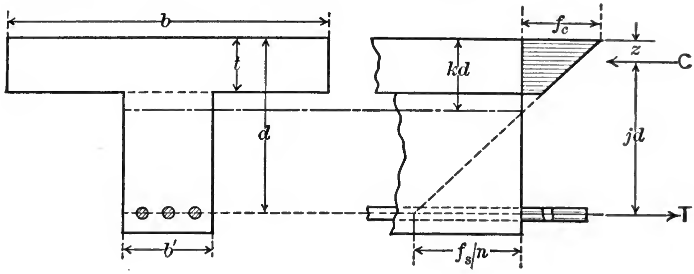


FIG. 20.

The following notation is employed in addition to that of Art. 53. (See Fig. 20.)

b = width of flange;
d = effective depth of beam;
b' = width of web;
t = thickness of flange;
z = depth of compression resultant below top of flange,
p = steel ratio $= A/bd$.

77. Case I. Compression in Web Neglected.—The amount of the compression in the web is commonly small compared to that in the flange and will be neglected in the analysis of this article. The formulas are thereby greatly simplified and the resulting error is generally very small. To provide for designs in which the web is very large as compared to the flange, formulas which take account of web compression are given in Art. 80.

78. *Neutral Axis and Arm of Resisting Couple.*—Just as in Art. 55, eq. (*a*),

$$\frac{f_s}{nf_c}=\frac{1-k}{k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

hence we have, in terms of f_s and f_c,

$$k=\frac{1}{1+f_s/nf_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Note that in terms of the unit stresses the position of the neutral axis is the same as in rectangular beams.

The average unit compressive stress in the flange is $\frac{1}{2}\left[f_c+f_c\left(1-\frac{t}{kd}\right)\right]=f_c\left(1-\frac{t}{2kd}\right)$, and the whole compression is $f_c\left(1-\frac{t}{2kd}\right)bt$. And since the whole tension and whole compression on the section are equal,

$$f_sA=f_c\left(1-\frac{t}{2kd}\right)bt. \quad \ldots \ldots \quad (b)$$

Eliminating f_s/f_c between equations (a) and (b) we get an equation which when solved for k gives

$$k=\frac{nA+\frac{1}{2}bt\cdot\frac{t}{d}}{nA+bt}. \quad \ldots \ldots \quad (2)$$

Substituting pbd for A we also derive the form

$$k=\frac{pn+\frac{1}{2}\left(\frac{t}{d}\right)^2}{pn+\frac{t}{d}}. \quad \ldots \ldots \quad (3)$$

The arm of the resisting couple is $d-z$ (see Fig. 20). The distance z is equal to the distance of the centroid of the shaded trapezoid from the top of the beam, that is,

$$z=\frac{3k-2\frac{t}{d}}{2k-\frac{t}{d}}\cdot\frac{t}{3}. \quad \ldots \ldots \quad (4)$$

We also have

$$jd=d-z, \quad \ldots \ldots \quad (5)$$

and, by substitution from (3) and (4) we have, in terms of t/d and p,

$$j=\frac{6-6\frac{t}{d}+2\left(\frac{t}{d}\right)^2+\left(\frac{t}{d}\right)^3/2pn}{6-3\frac{t}{d}}. \quad . \quad . \quad . \quad (6)$$

When $t/d=k$, this reduces to eq. (2), Art. 55.

The neutral axis will be at the junction of web and flange when $t/d=k$.

On Plate IV are plotted curves giving values of k and j for various values of p and of the ratio t/d. The value of n is taken at 15. This diagram, as well as eq. (6), shows that j is affected very little by changes in the amount of steel. The diagram also gives on the right-hand margin the values of f_s/f_c, corresponding to the various values of k as determined from eq. (1). The curves for k and j end at points where $k=t/d$. They become horizontal at these points and the values of k are equal to those for rectangular beams. (See Fig. 9, Art. 55.)

In investigating a given beam, k and j should be found from the known values of p, t, and d, using eq. (3) or Plate IV. Eq. (1) involving fibre stresses, cannot be used as the actual relation of these stresses is not known, but from Plate IV, the true ratio of fibre stresses can be determined from the right-hand margin.

79. *Resisting Moment and Working Stresses.*—As already stated, the total compression is $f_c\left(1-\frac{t}{2kd}\right)bt$, and the total tension is f_sA; hence, the resisting moment in terms of f_c and f_s is given by the formulas

$$\left.\begin{aligned} M_s&=f_sA\cdot jd \\ M_c&=f_c\left(1-\frac{t}{2kd}\right)bt\cdot jd \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If f_c and f_s are merely the permissible values of stress, then the calculated values of M_c and M_s will not in general be equal; the safe value is evidently the smaller.

The unit stresses, f_s and f_c, produced by a certain bending moment M in a given beam can be computed by solving (7) for f_s and f_c, or from

$$\left.\begin{aligned} C=T=\frac{M}{jd};\quad f_s=\frac{T}{A};\\ f_c=\frac{f_s}{n}\cdot\frac{k}{1-k}=\frac{f_sp}{\left(1-\frac{t}{2kd}\right)\frac{t}{d}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

After f_s is calculated, f_c can most readily be found from the ratio f_s/f_c given in Plate IV.

Approximate formulas corresponding to (7) and (8) can be established as follows: From the stress diagram in Fig. 20, it is plain that the arm of the resisting couple is never as small as $d-\frac{1}{2}t$, and that the average unit compressive stress is never as small as $\frac{1}{2}f_c$, except when the neutral axis is at the top of the web. Using these limiting values as approximations for the true ones, we have as substitutes for (7) and (8)

$$\left.\begin{aligned} M_s=Af_s(d-\tfrac{1}{2}t)\\ M_c=\tfrac{1}{2}f_cbt(d-\tfrac{1}{2}t) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$C=T=\frac{M}{d-\frac{1}{2}t},\quad f_s=\frac{T}{A},\quad f_c=\frac{2C}{bt}. \quad . \quad . \quad . \quad . \quad (10)$$

The errors involved in these approximations are on the side of safety, for (9) gives values smaller than (7), and (10) larger ones than (8). Satisfactory approximate results may also be reached by assuming a fixed value of $\frac{7}{8}d$ for the arm of the resisting couple jd.

80. Case II. Compression in Web Not Neglected.—When the web is very large compared to the flange it may be desirable to use more exact formulas than those already given. In this case the formulas for the position of neutral axis, arm of resisting couple, and moment of resistance become as follows:

$$kd=\sqrt{\frac{2ndA+(b-b')t^2}{b'}+\left(\frac{nA+(b-b')t}{b'}\right)^2}-\frac{nA+(b-b')t}{b'} \quad (11)$$

$$z = kd - \frac{2}{3}\frac{b(kd)^3 - (b-b')(kd-t)^3}{b(kd)^2 - (b-b')(kd-t)^2}, \quad \ldots \ldots \quad (12)$$

$$jd = d - z; \quad \ldots \ldots \quad (13)$$

$$\left.\begin{aligned} M_s &= f_s A \cdot jd \\ M_c &= \frac{f_c}{2kd}[b(kd)^2 - (b-b')(kd-t)^2]jd \end{aligned}\right\} \quad \ldots \ldots \quad (14)$$

Equations (14) also give f_s and f_c for given values of M.

81. Diagrams of M/bd^2 for Use in Designing.—By reason of the additional variable (the flange thickness) involved in formulas relating to T-beams as compared to rectangular beams, it is not possible to arrange so simple a graphical solution of the resisting moment as is done by means of Plates I–III, Arts. 60, 65. Assuming, however, a single value of n, the values of the resisting moment or coefficients of resistance may readily be represented graphically. It will be convenient to consider as variables the values of f_s, f_c, and the ratio t/d, or thickness of slab to the effective depth of the beam. From eq. (7) we may write

$$\frac{M}{bd^2} = f_c\left(1 - \frac{t}{2kd}\right) \cdot \frac{t}{d} \cdot j. \quad \ldots \ldots \quad (15)$$

In this equation k and j are functions of f_s and f_c, as appears from eqs. (1), (4), and (5). Plates V–VIII are plotted from this equation, assuming $n = 15$ in all cases. Each plate contains values of M/bd^2 for a certain value of f_s and for various values of f_c and t/d. On the same diagram are also given values of k and values of j. The former are given by the dotted curve in the right-hand part of the diagram, and the value corresponding to a given value of f_c is to be read off on the axis for t/d. This value of k, is in fact, the value of t/d which brings the neutral axis just to the lower surface of the flange. The use of the diagrams will be illustrated by the solution of examples.

82. Problems of Design.—In practice, various forms of problems will arise: (*a*) The dimensions may be given, to find the safe resisting moment of the beam or the stresses in the steel and concrete under a given load; (*b*) the dimensions of the flange may be given, together with the loading and specified working stresses, to determine suitable web dimensions and steel area; (*c*) the loading and working stresses may be given, to determine suitable proportions for the entire beam.

(*a*) Where all the dimensions are given, the value of k and j are found from eqs. (3) and (6), or from Plate IV, and thence the values of the moment of resistance from eqs. (7), or the fibre stresses from (7) or (8). If the value of k is found to be less than t/d, then the formulas for rectangular beams apply.

(*b*) Generally the flange has been predetermined as it is usually formed by a portion of a floor slab which is already designed. A suitable web must then be determined, together with the necessary amount of steel; and finally the fibre stress in the concrete must be calculated to ascertain if it is within the specified working limit. The depth and width of web are selected with reference to shearing strength, space for the necessary rods and other considerations, as fully explained in subsequent articles. The depth having been selected, the value of j is estimated and the amount of steel, A, approximately determined by eq. (7). The amount of steel being known, the value of j can be accurately determined by eq. (6) or Plate IV, and then, if necessary, the value of A corrected by eq. (7). The value of k should also be found from eq. (2) or Plate IV, in order to ascertain if the beam is a T-beam as assumed. The stress in the concrete is then found from eq. (8) or Plate IV.

In estimating the value of j use Plate IV, or a value of $\frac{7}{8}$ may be assumed, as for rectangular beams.

(*c*) When all parts of the beam are to be selected on the basis of given working stresses it is convenient to first select suitable proportions for the web, as in Case (*b*). A flange

thickness is then assumed such as to give satisfactory proportions between t and d. The value of t/d is then known and k and j can be determined from (1), (4), and (5). The area of steel and the breadth of flange is then found from eq. (7). The smaller the value of t the smaller will be the flange area required, but too slender proportions are to be avoided, as explained in Chapter V.

Plates V–VIII are also directly applicable to this case. From the given values of working stresses and the assumed value of t/d, the value of M/bd^2 is at once obtained. From this bd^2 is calculated and convenient values selected for d and b. The value of t is given from the assumed ratio t/d, and the amount of steel required from eq. (7) or less accurately from Plate IV, using the known value of f_s/f_c and t/d.

83. Examples.—(1) A T-beam has the following dimensions: $b=48$ inches, $t=6$ inches, $d=22$ inches, and $b'=10$ inches; the steel consists of six $\frac{3}{4}$-inch rods. If the working strengths of steel and concrete are 15,000 and 600 lbs/in², respectively, and $n=15$, what is the safe resisting moment of the beam?

Solution. The area of the steel is 2.65 in², and $p=2.65/(48\times22)=0.0025$. From eq. (2), Art. 78, we find that k is less than t/d, hence the neutral axis is in the flange and the beam is to be calculated as a rectangular beam. From Fig. 9 $j=0.92$; hence (Art. 56)

$$M_s=(15{,}000\times2.65)(0.92\times22)=806{,}000 \text{ in-lbs.},$$

and $$M_c=300(5.3\times48)(0.92\times22)=1{,}545{,}000 \text{ in-lbs.}$$

The safe resisting moment hence depends on the steel, as it usually does in T-beams.

(2) Change t of the preceding example to 4 inches and find the safe resisting moment.

Solution. Equation (2) gives $k=0.247$ and (4) $z=1.61$ inches. From (7)

$$M_s=15{,}000\times2.65(22-1.61)=812{,}000 \text{ in-lbs.},$$

and $$M_c=(600/5.44)3.44(48\times4)(22-1.61)=1{,}485{,}000 \text{ in-lbs.}$$

The values of k and j may also be found from Plate IV.

The approximate formulas (9) give $M_s=716{,}000$ and $M_c=1{,}152{,}000$ in-lbs.

(3) Suppose that the diameter of the rods in example (1) is 1 inch,

and that the beam is subjected to a bending moment of 1,250,000 in-lbs. Compute the working stresses in the steel and concrete.

Solution. Equation (2) gives $k=0.306$ and $kd=6.73$. Equation (4) gives $z=2.22$ inch, and (8)

$$f_s=\frac{1,250,000}{(22-2.22)4.71}=13,400 \text{ lbs/in}^2,$$

and
$$f_c=\frac{13,400}{15}\,\frac{0.306}{1-0.306}=395 \text{ lbs/in}^2.$$

The approximate formulas (10) give $f_s=13,960$ and $f_c=457$ lbs/in².

(4) The flange of a T-beam is 24 inches wide and 4 inches thick. The beam is to sustain a bending moment of 480,000 in-lbs., the working strengths of steel and concrete being respectively 15,000 and 500 lbs/in². What depth of beam and amount of steel will answer?

Solution. We will try $d=16$ inches. Assume $j=0.9$. Then eq. (7) gives $A=2.22$ in² and hence $p=\dfrac{2.22}{24\times16}=0.00577$. Then (6) gives $j=0.896$ and the corrected value of $A=2.23$ in² and $p=0.0058$. Eq. (3) gives $k=0.351$ and shows that the neutral axis is in the web as assumed. The stress in the concrete is found from (8) to be $\dfrac{15\,000\times0.351}{15\times.649}=541$ lbs/in². Since the maximum allowable value for f_c is 500 lbs/in² the beam must be made deeper.

We will try $d=18$ inches and proceed as before. Assuming $j=0.89$, eq. (7) gives $A=2.00$ in². Then $p=\dfrac{2.00}{24\times18}=0.00462$ and j from eq. (6) becomes 0.91 and the corrected value of A 1.95 in². Eq. (3) gives $k=0.325$ and the stress in the concrete is found to be 480 lbs/in² which is permissible.

(5) Let it be required to design a T-beam to sustain a bending moment of 480,000 in-lbs., the working stresses to be 15,000 and 600 lbs/in², respectively.

Solution. The same depth of web and thickness of flange will be assumed as in (4), as these proportions are reasonable. The amount of steel and width of flange are to be determined. From (1) we find $k=1\div(15,000/9000+1)=0.375$, and from (4) $z=1.72$ and $jd=18.0-1.72=16.28$ inches. Then from (7), $A=1.97$ in² and $b=17.5$ inches.

The preceding examples will now be solved as far as practicable by the use of Plates V–VIII.

(1) Plate IV can be used to determine whether the neutral axis is in flange or in the web. We first find the value of $t/d=0.273$ at the bottom of the diagram and then trace vertically to the point where $p=0.0025$.

The value of k given on the left of the diagram is less than t/d, and the beam must be solved as a rectangular beam.

(2) Proceeding as in the above example, the intersection of $p=0.0025$ and $t/d=0.182$ gives $k=0.245$, and to the right $f_s/f_c=47$. From the same diagram j is also found to be 0.93. The ratio of f_s to f_c for permissible values is $15{,}000 \div 600=25$ hence it is obvious that the steel must reach 15,000 lbs/in² before the concrete reaches 600 lbs/in². The resisting moment depends therefore upon the steel and must be calculated by the formula.

(3) Plate IV can again be used to determine the position of the neutral axis, the values of k, j and f_s/f_c. Entering the diagram with $p=\dfrac{6\times.785}{48\times22}=0.00445$ and $t/d=0.273$, we find $k=0.310$, $f_s/f_c=34$ and $j=0.90$, and that the neutral axis is in the web. The unit stresses corresponding to a bending moment of 1,250,000 must then be calculated by formula. Eq. (7) gives

$$f_s=\frac{M}{Ajd}=\frac{1{,}250{,}000}{4.710\times0.90\times22}=13{,}400 \text{ lbs/in}^2.$$

Also, $$f_c=f_s/34=13{,}400/34=395 \text{ lbs/in}^2.$$

(4) Use Plate VI. The value of $M/bd^2=480{,}000/(24\times18^2)=61.6$. For this value of M/bd^2 and for $t/d=4/18$, we find from the diagram $f_c=$about 470 lbs/in² and $j=.91$. Then, as before, $A=480{,}000/(15{,}000\times .91\times18)=1.95$ in².

(5) Use Plate VI. For $f_c=600$ and $t/d=4/18$, we find $M/bd^2=85$, whence $b=480{,}000/(85\times18^2)=17.4$ inches; also, $j=.90$ and $A=1.97$ in².

(6) Using the same depth of beam as in Ex. (5) what will be the effect on the amount of concrete and steel of changing the flange thickness to 3 inches and to 5 inches? From Plate VI we find, for $t/d=3/18$, $M/bd^2=74$, whence $b=20.0$ inches, $j=.92$ and $A=1.93$ in². For $t/d=5/18$, $M/bd^2=93$, $b=16.0$ inches, $j=.89$ and $A=2.00$ in². The volumes of the concrete in the flange are, for the flange thicknesses of 3, 4 and 5 inches, respectively, 60, 70, and 80 sq. in. The steel areas vary but little.

BEAMS REINFORCED FOR COMPRESSION.

84. Assumptions and Notations.—Generally speaking, it is more economical to carry compressive stresses by concrete than by steel, but, under certain circumstances, it is desirable to reinforce the compressive side of a beam so as to reduce the stresses in the concrete. Flexure formulas for such a case are here developed.

The compression in the concrete is assumed to follow the linear law and the tension in it is neglected; the formulas then apply to working conditions only. In addition to the notation already adopted let

A' = cross-sectional area of the compressive reinforcement;
p' = steel ratio for the compressive reinforcement $= A'/bd$;
f_s' = unit stress in the compressive reinforcement;
C' = whole stress in the compressive reinforcement;
d' = distance from the compressive face of the beam to the plane of the compressive reinforcement;
z = distance from the compressive face to the resultant compression, $C+C'$, on the section of the beam.

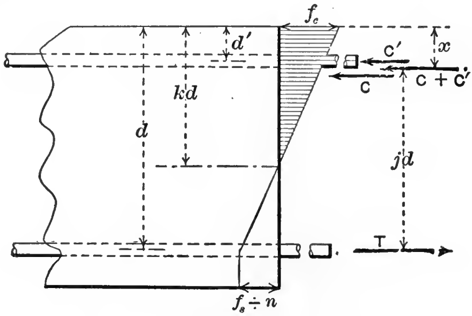


FIG. 21

85. Derivation of Formulas.—*Neutral Axis and Arm of Resisting Couple.*—From the stress diagram (Fig. 21) it appears that $f_s/nf_c = (d-kd)/kd$, or

$$f_s = n\frac{1-k}{k}f_c. \quad \ldots \quad (1)$$

Similarly, $f_s'/nf_c = (kd-d')/kd$, or

$$f_s' = n\frac{k-d'/d}{k}f_c. \quad \ldots \quad (2)$$

For simple flexure, the whole tension T and whole compression $C+C'$ are equal, hence

$$f_sA = \tfrac{1}{2}f_cbkd + f_s'A'. \quad \ldots \quad (a)$$

Inserting the values of f_s and f_s' from (1) and (2) in (a) gives an equation which may be written thus:

$$k^2+2n(p+p')k=2n(p+p'd'/d), \quad . \quad . \quad . \quad . \quad (3)$$

and from this the neutral axis of a given section can be located.

The arm of the resisting couple is the distance between T (see Fig. 21) and the resultant of the compressions C and C'. It follows from the principle of moments and the law of distribution of stress respectively that

$$z=\frac{\frac{1}{3}k+d'C'/dC}{1+C'/C}d, \quad \text{and} \quad \frac{C'}{C}=\frac{2p'n(k-d'/d)}{k^2},$$

from which z can be computed for any given section. Finally the arm $jd=d-z$ or

$$j=(1-z/d). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Figs. 22 and 23 give values of k and j for several values of p and for all values of p' up to 2%; n is taken at 15, and d'/d as .10 and .15 respectively.

86. *Resisting Moment and Working Stresses.*—If the tensile reinforcement is low, the resisting moment depends upon it, and is given by

$$M_s=f_sAjd=f_spjbd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the compressive reinforcement is low, the resisting moment depends upon it and the concrete, and is given by

$$M_c=\tfrac{1}{2}f_ck(1-\tfrac{1}{3}k)bd^2+f_s'p'bd(d-d');$$

but f_s' bears a certain relation to f_c (see eq. 2), which inserted in the preceding equation gives finally

$$M_c=[k(\tfrac{1}{2}-\tfrac{1}{6}k)+np'(k-d'/d)(1-d'/d)/k]f_cbd^2. \quad . \quad . \quad (6)$$

The unit fibre stress in the tensile steel produced by any bending moment M can be computed from

$$f_s=\frac{M/jd}{A}=\frac{M}{pjbd^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

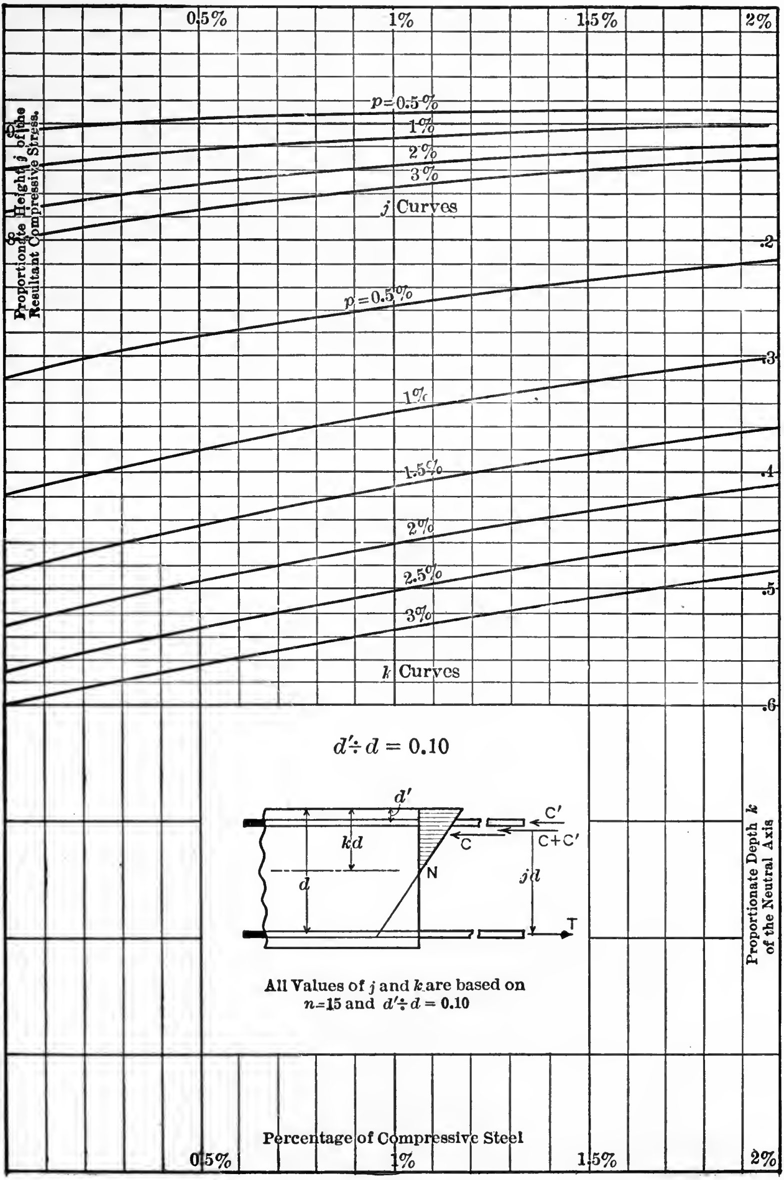

0.5%
1%
1.5%
2%
$p = 0.5\%$
1%
2%
3%
j Curves
Proportionate Height j of the Resultant Compressive Stress.
.9
.8
$p = 0.5\%$
1%
1.5%
2%
2.5%
3%
k Curves
.2
.3
.4
.5
.6
$d' \div d = 0.10$
d'
kd
d
C'
C
C+C'
N
jd
T
All Values of j and k are based on
$n = 15$ and $d' \div d = 0.10$
Proportionate Depth k of the Neutral Axis
Percentage of Compressive Steel

FIG. 22.

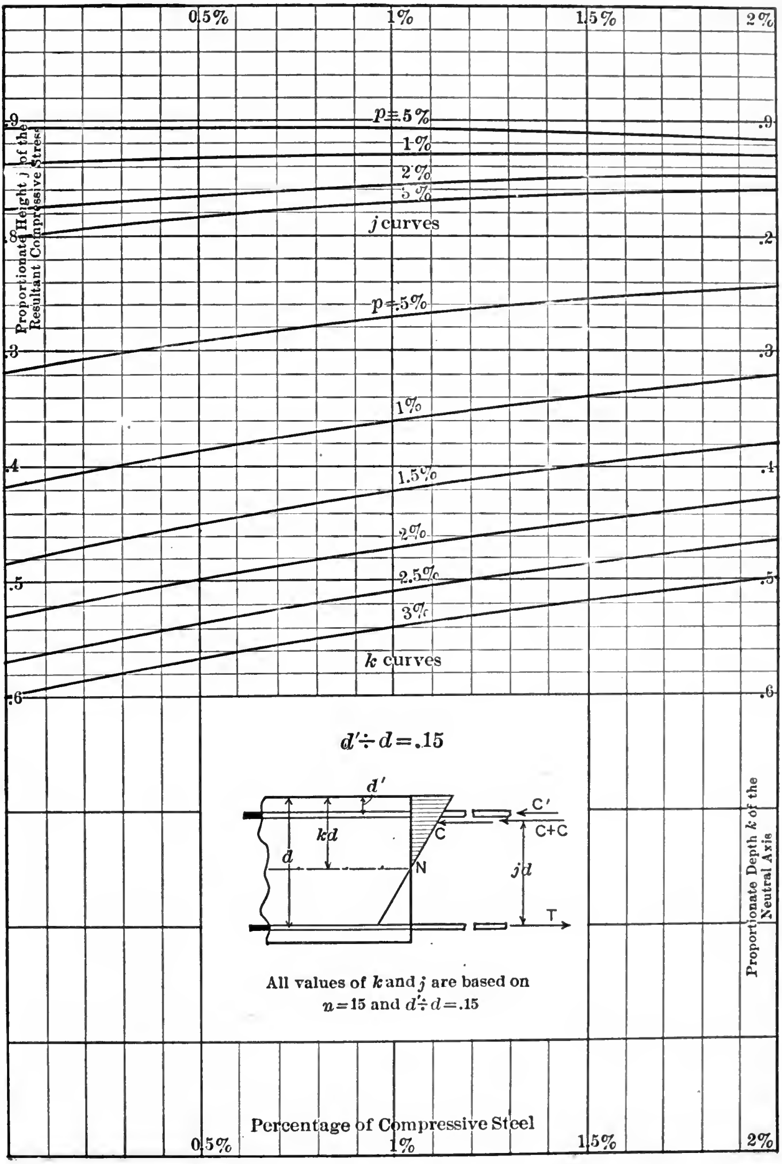

0.5%
1%
1.5%
2%
Proportionate Height j of the Resultant Compressive Stress
p=.5%
1%
2%
3%
j curves
p=.5%
1%
1.5%
2%
2.5%
3%
k curves
.9
.8
.2
.3
.4
.5
.6
$d' \div d = .15$
d'
kd
d
C'
C+C
C
N
jd
T
All values of k and j are based on
$n = 15$ and $d' \div d = .15$
Proportionate Depth k of the Neutral Axis
Percentage of Compressive Steel

FIG. 23.

and those in the concrete and compressive steel from f_s and equations (1) and (2) respectively.

It will be found that the stress f_s' need never be computed, for it will always be less than nf_c. Taking f_c at the high value of 700 lbs/in² and $n=15$, then f_s' will be less than $15\times700=$ 10,500 lbs/in².

87. *Determination of Amount of Compressive Reinforcement.*—This problem presents itself as follows: From the circumstances of the case, the beam needs so much tensile steel that the compressive concrete, if unreinforced, would be stressed too high, and it is necessary to employ compressive reinforcement to reduce the stress in the concrete; the percentage of reinforcement necessary to lower the stress a certain amount is desired.

An explicit formula for this percentage is too cumbersome for practical use, but a diagram (Plate IX) can be constructed from which the desired quantity can be easily determined. The construction of such a diagram will now be explained.

Let f_s and f_c denote the unit stress in the tensile steel and the concrete respectively, kd the depth of the neutral axis, and jd the arm of the resisting couple $C-T$, when there is no compressive reinforcement (see Fig. 8); also let f_s', f_c', $k'd$, and $j'd$ denote the same quantities when there is compressive reinforcement. Then

$$f_c=\frac{f_s kd}{n(d-kd)}=\frac{Mk}{jdAn(1-k)},$$

and

$$f_c'=\frac{f_s'k'd}{n(d-k'd)}=\frac{Mk'}{j'dAn(1-k')}.$$

From these the *relative* reduction in f_c due to the addition of compressive steel is found to be

$$\frac{f_c-f_c'}{f_c}=1-\frac{j}{j'}\frac{k'}{k}\frac{1-k}{1-k'}. \qquad (8)$$

Since j and k depend on p, and j' and k' on p and p', the equation furnishes the relation between relative reduction in con-

crete stress and the percentages of steel. The relative reduction $(f_c - f_c')/f_c$ depends largely on the percentage of compressive steel and for a given value of this percentage the reduction is practically the same for all ordinary percentages of tensile steel (from $\frac{1}{2}$ to 3%). Plate IX gives values of this reduction for different values of compressive steel from 0 to 2%. For the upper group of curves the ratio d'/d is taken at .15 and for the lower group at .10.

Addition of compressive steel reduces the stress in the tensile steel. The relative amount of this reduction is given by

$$\frac{f_s - f_s'}{f_s} = 1 - \frac{j}{j'}. \quad \ldots \ldots \quad (9)$$

The lower groups of curves (Plate IX) gives this reduction in per cent (right-hand margin) for different percentages of tensile and compressive steels as noted. For illustration of the use of this diagram, see example (3) following.

Examples.—(1) A beam of which $b = 12$ in., $d = 18$ in., and $d'/d = {}^1/_{10}$ has $2\frac{1}{2}$% of tensile steel and 1% of compressive. If the working strengths of steel and concrete are 15,000 and 600 lbs/in² respectively, what is the safe resisting moment of the beam?

Solution. From Fig. 22, $k = 0.5$ and $j = 0.85$; therefore

$$M_s = 15{,}000 \times 0.025 \times 0.85 \times 12 \times 18^2 = 1{,}238{,}000 \text{ in-lbs.},$$

and

$$M_c = (0.5 \times 0.417 + 15 \times 0.01 \times 0.4 \times 0.9/0.5)600 \times 12 \times 18^2 = 736{,}000 \text{ in-lbs.},$$

which is the safe resisting moment.

(2) Suppose that the beam of the preceding example were subjected to a bending moment of 1,000,000 in-lbs. What are the working stresses f_c, f_s, and f_s'?

Solution. As in example (1), $k = 0.5$ and $j = 0.85$; therefore (see eq. 7) $f_s = 1{,}000{,}000/0.025 \times 0.85 \times 12 \times 18^2 = 12{,}100$ lbs/in². From equation (1), $f_c = (12{,}100 \times 0.5) \div (15 \times 0.5) = 810$ lbs/in², and from equation (2), $f_s' = 15(0.4/0.5)810 = 9720$ lbs in².

(3) In a certain design of a beam it is necessary to use 2.5% of tensile steel and this would result in a stress of 1200 lbs/in² in the concrete; it is necessary to reduce this to 900 by adding compressive steel. How much additional steel is required? Assume $d'/d = .10$.

Solution. (See Plate IX.) The desired reduction of the compressive stress is 25%. We find this value at the left side of the diagram, then trace horizontally to the concrete curve, and then down to the lower margin, reading there 0.9%, the required quantity. From the last point we trace up to the 2.5% steel curve and then to the right margin, where we note about 4.5% reduction in tensile steel stress due to 0.9% compressive steel.

FLEXURE AND DIRECT STRESS.

88. General Conditions.—In the preceding cases of flexure it has been assumed that the resultant of the external forces acting on one side of the section was parallel to the section (perpendicular to the axis), producing what is commonly called pure flexure or bending moment only. When this resultant is not parallel to the section there will exist both flexure and direct stress except in the case of a centrally applied load producing direct stress only. In the case of reinforced concrete we are concerned with combined compression and bending, the most common case being the concrete arch where both compression and bending exist at most sections.

89. Notation.—In the analysis of beams under combined stresses, especially arches, it is convenient to consider the external forces on one side of the section combined into a resultant R (Fig. 24) whose direction and line of action are known. If O is the centroid of the section AB, the distance e from O to the point where the resultant R cuts the section is the eccentric distance. Resolving R in directions normal and parallel to the section gives the component N, which is the direct stress or thrust, and V, which is the shear acting on the section. The bending moment, M, is equal to Ra or Ne. It will be assumed that the beam is reinforced near both top and bottom faces.

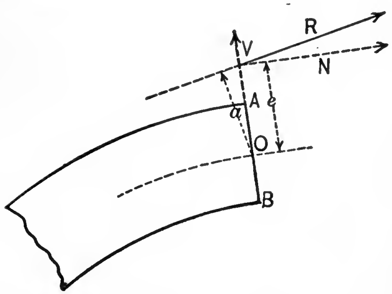


FIG. 24.

In addition to the notation already adopted, the following will be used (Figs. 25 and 26). For convenience, the face of the beam most highly stressed is called the "compression

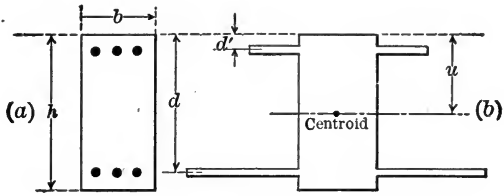


FIG. 25.

face" and the opposite face is called the "tension face"; it may be stressed either in tension or compression.

R = resultant force acting on the section;

N = component of R normal to section;

e = eccentric distance of R, e/h = eccentricity;

M = bending moment = Ne;

A' = area of steel near compressive face;

$p' = A'/bh$;

A = area of steel near tension face;

$p = A/bh$;

d' = distance of compressive steel from face;

u = distance from compressive face to centroid of transformed section;

h = whole height of section;

a = distance from steel to center of section for symmetrical reinforcement;

A_t = area of transformed section;

I_c = moment of inertia of concrete about centroidal axis of transformed section;

I_s = moment of inertia of steel about centroidal axis of transformed section;

I_t = moment of inertia of transformed section;

f_c = maximum compressive fibre stress in concrete;

f_c' = maximum tensile fibre stress in concrete;

f_s' = stress in steel near compressive face;

f_s = stress in steel near tension face.

90. The Transformed Section.—If the steel areas in a reinforced beam be replaced by concrete in the same plane and having n times the area of the steel the resulting section is called the "transformer section." Such an ideal section can then be used in place of the actual combination of steel and concrete, for purpose of stress analysis, in the same manner as a beam of any homogeneous material, so long as the stresses

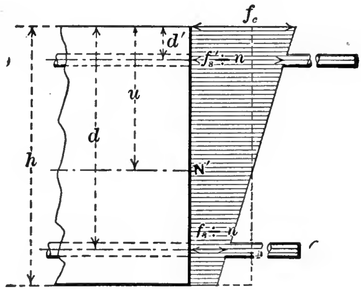


FIG. 26.

are all compressive or if the tensile stresses in the concrete are so small as to be permissible.

Thus, if Fig. 25(a) represents an actual section, 25(b) represents the transformed section, the areas of the upper and lower wings being respectively n times the areas of the upper and lower reinforcements.

Referring to Fig. 25 it will readily be seen that

$$A_t = bh + n(A + A'), \quad I_t = I_c + nI_s, \quad \ldots \quad (1)$$

$$u = \frac{h/2 + npd + np'd'}{1 + np + np'}, \quad \ldots \quad (2)$$

$$I_c = \tfrac{1}{3}b[u^3 + (h-u)^3] \quad \text{and} \quad I_s = A(d-u)^2 + A'(u-d')^2. \quad (3)$$

If the reinforcement is symmetrical, then $u = h/2$ and

$$I_c = \tfrac{1}{12}bh^3 \quad \text{and} \quad I_s = 2A(\tfrac{1}{2}h - d')^2. \quad \ldots \quad (3)'$$

91. Cases to be Considered.—If the eccentric distance e is within certain limits, then the stress on the section is wholly compressive (Figs. 26 and 27), but if it exceeds this limit, there will be tensile stress on the section (Fig. 28). If it be assumed that the concrete takes no tension then the analysis for these two cases is quite different.

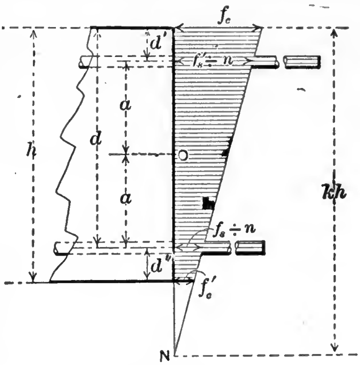


FIG. 27.

Whether a given R will produce fibre stress falling under case (1) or (2) depends on the eccentricity of R, the relative amounts of steel and concrete at the section and on n. If the reinforcement is symmetrical, steel imbedded a depth equal to 1/10 the whole depth of beam, and n is 15, then for eccentricities *lower* than those given in the table, case (1) obtains, and for *higher*, case (2).

$p=$	0%	$\frac{1}{2}$%	1%	$1\frac{1}{2}$%	2%
$e/h=$	$\frac{1}{6}$	0.187	0.202	0.214	0.244

In practice the problem will usually be to find the maximum fibre stress f_c for a given design and a given M. In the arch, for example, a design is selected by the use of empirical formulas or by comparison with previous designs, the moments and thrusts (M and N), determined, and from these the stresses in the concrete. In this analysis, therefore, we are principally concerned with the development of formulas from which the fibre stress can be determined for a given beam and given values of M and N.

92. Case I. The Fibre Stress is Wholly Compressive.—There are two methods of treatment.

(a) The unit fibre stress in the concrete can be computed just as though the beam were homogeneous, but the trans-

formed section must be used. The unit stresses in the steel will be n times those in the concrete in the planes of the reinforcements respectively. Thus, the unit direct stress in the concrete is N/A_t; the unit flexural stress in the concrete on the compression side is Mu/I_t; that in the concrete adjoining the reinforcement on the same side is $M(u-d')/I_t$; and that in the concrete adjoining the other reinforcement is $M(d-u)I_t$. The combined unit stresses are:

$$f_c=\frac{N}{A_t}+\frac{Mu}{I_t}, \quad \ldots \quad (4)$$

$$f_s'=n\frac{N}{A_t}+\frac{nM(u-d')}{I_t}, \quad \ldots \quad (5)$$

$$f_s=n\frac{N}{A_t}-\frac{nM(d-u)}{I_t}. \quad \ldots \quad (6)$$

These equations—and the stress diagram, Fig. 26—show that f_s is always less than f_s', and f_s' is always less than nf_c; hence the unit stresses in both steel reinforcements will always be safe if f_c is a safe value.

(b) The method employed for simple flexure, suitably modified, leads to formulas not involving the transformed section, as will now be explained.

From the stress diagram (Fig. 27) it will be seen that

$$f_s'=nf_c(1-d'/kh), \quad \ldots \quad (7)$$

$$f_s=nf_c(1-d/kh), \quad \ldots \quad (8)$$

and

$$f_c'=f_c(1-1/k). \quad \ldots \quad (9)$$

From the condition that the resultant fibre stress equals N,

$$\tfrac{1}{2}(f_c+f_c')bh+f_s'A'+f_sA=N;$$

and from the condition that the moment of the total fibre stress about the centroidal axis equals M,

$$\tfrac{1}{2}(f_c+f_c')bh\frac{h}{6(2k-1)}+f_s'A'\left(\frac{h}{2}-d'\right)-f_sA\left(\frac{h}{2}-d'\right)=M.$$

From these equations it is possible to compute the unit fibre stresses f_c, f_s, and f_s' in a given case.

When the reinforcement is symmetrical the equations simplify greatly, and they lead to the following formula:

$$12k(1+2np)e/h=1+24npa^2/h^2+6(1+2np)e/h; \quad . \quad (10)$$

they also give the following formula for f_c or M:

$$\frac{M}{bh^2f_c}=\frac{1}{12k}(1+24npa^2/h^2). \quad . \quad . \quad . \quad . \quad (11)$$

Plate X gives values of $1/k$ for different values of the eccentricity, e/h, and different values of p. The value of d'/d is assumed to be 1/10 and $n=15$. The value of k being known the value of f_c for any given moment, or the allowable moment for a given value of f_c, can be obtained from (11). This diagram is applicable only to symmetrical reinforcement. For unsymmetrical reinforcement the best method is by use of the transformed section, eq. (4).

93. Case II. There is Some Tension at the Section.—(*a*) If the tension in the concrete is so small as to be permissible, and this tension is taken account of in the computations, then the unit fibre stresses in the concrete and steel, if reinforcement is present, may be computed by the method explained under Case I.*

The combined unit stress in the remote tensile fibre is given by

$$f_c'=\frac{M(h-u)}{I_t}-\frac{N}{A_t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

* It is assumed that the linear law of variations of the unit flexural stresses holds for the tension as well as compression.

and f_s as given by (6) is compressive or tensile according as its value is positive or negative.

(*b*) If the tensile stresses are so high that it is advisable to neglect the tension in the concrete, then a method similar to that used heretofore in simple flexure is simplest. The transformed section is not used. O (Fig. 28) denotes a horizontal

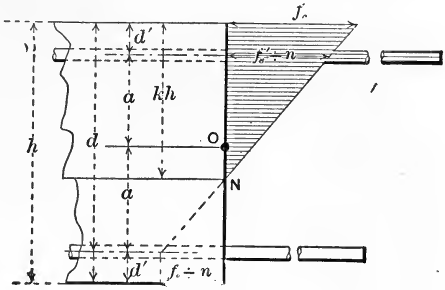


FIG. 28.

axis at mid-depth of the beam, M the moment sum of all the external forces on one side of the section with respect to that axis, and N, as before, the algebraic sum of the components of those forces perpendicular to the section. From the stress diagram, it follows that

$$f_s = nf_c\left(\frac{d}{kh} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$f_s' = nf_c\left(1 - \frac{d'}{kh}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Since the resultant fibre stress equals N,

$$\tfrac{1}{2}f_c bkh + f_s'A' - f_sA = N,$$

and since the moment of the fibre stress about the horizontal axis through O equals M,

$$\tfrac{1}{2}f_c bkh\left(\frac{h}{2} - \frac{kh}{3}\right) + f_s'A'\left(\frac{h}{2} - d'\right) + f_sA\left(d - \frac{h}{2}\right) = M.$$

From these four equations k, f_c, f_s, and f_s' can be determined for a given section, reinforcement, M, and N.

If the reinforcement is symmetrical, then the equations simplify. The value of k is given by

$$k^3-3\left(\frac{1}{2}-\frac{e}{h}\right)k^2+12np\frac{e}{h}k=6np\left(\frac{e}{h}+2\frac{a^2}{h^2}\right). \quad . \quad . \quad (15)$$

The greatest unit compressive fibre stress in the concrete is given by

$$\frac{M}{bh^2f_c}=\frac{1}{12}k(3-2k)+\frac{2pn}{k}\frac{a^2}{h^2}, \quad . \quad . \quad . \quad . \quad (16)$$

and the unit stresses in the steel are given by (7) and (8). From (7), or the stress diagram, it is plain that f_s' is less than nf_c even for unsymmetrical reinforcements.

Plate XI gives values of k for various values of e/h and p, for symmetrical reinforcement and $d'/d=1/10$. $n=15$.

94. **Diagrams of M/bh^2f_c for Use in Designing.**—To facilitate the application of eq. (11) (Case I), and eq. (16) (Case II), Plates XII and XIII, Chapter XIV, have been constructed.

In the first diagram, values of the eccentricity, e/h, are given at the upper and lower margins; the ordinates from the lower margin to any curve are values of $(1+24npa^2/h^2)/12k$ (see equation 11), and hence of M/bh^2f_c, for the value p marked on that curve. Thus when $e/h=0.1$ and $p=1\%$, $M/bh^2f_c=0.087$.

The dotted portions of the curves correspond to eccentricities which involve small tensile stress in the concrete and belong strictly to Case II. The values of the unit tensile stress f_c' can be calculated from equation (12) or from

$$\frac{f_c'}{f_c}=\frac{h-kh}{kh}=\frac{1}{k}-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$1/k$ being obtained from equation (10), or from an extension of the appropriate curve in Plate X.

In the second diagram, also, values of the eccentricity e/h are given at the upper and lower margins; the ordinates from the lower margin to any solid curve are values of

$\frac{1}{12}k(3-2k)+2pna^2/kh^2$ (see equation 16), and hence of M/bh^2f_c, for the value of p marked on that curve. Thus when $e/h=1$ and $p=1\%$, $M/bh^2f_c=0.187$.

The dotted curves in the second diagram enable one to estimate the ratio of the unit stress in the tensile steel to that in the concrete, f_s/f_c, for most eccentricities and percentages of steel. Thus when $e/h=1$ and $p=0.5\%$, we find $e/h=1$ at the top or bottom and then trace vertically to the 0.5% curve and note the point of intersection. This point falls between the curves $f_s/f_c=20$ and 25, and the ratio is about 21. For values of e/h and p, which bring the "point" to the left of the line $f_s/f_c=15$, f_s will be less than $15f_c$, and hence less than the working strength of steel for all ordinary allowable values of f_c. No similar curves for f_s/f_c appear on the first diagram because that ratio is always less than 15, and hence the unit stresses in the steel (both upper and lower) are within safe values for Case I, if f_c is safe.

95. Examples.—In the usual problem the value of the resultant R is given and its eccentricity, from which the bending moment M is readily found. Then, with these quantities, it is required to determine the stresses in a given beam with known reinforcement or to determine the reinforcement required to keep the concrete stresses within specified limits. As explained in Art. 88 the usual case is the arch, but the analysis of the stresses is the same for any beam where the value of R and its position are known. The determination of the thrust and moment in the arch is fully discussed in Chap. X.

It is assumed in the following examples that the steel is imbedded a depth of one-tenth the total height of the beam, and that $n=15$, so that Plates XII and XIII apply.

(1) A beam is 12 inches wide, 30 inches high, and contains 1% of steel above and an equal percentage below. At a particular section, the resultant R is 80,000 lbs., its inclination to the axis of the beam is 5°, and its eccentric distance is 4.5 inches. Compute the unit fibre stresses in the concrete and steel (f_c, f_s, and f_s').

Solution. The eccentricity is $e/h=0.15$, and $M=80{,}000 \cos 5° \times 4.5=$

358,650 in-lbs. The beam falls under Case I because this eccentricity gives a " point " on the 1% curve of Plate XII, but not on that of Plate XIII. Tracing horizontally from the point we read $M/bh^2f_c=0.112$; hence

$$f_c=\frac{358,650}{12\times30^2\times0.112}=297 \text{ lbs/in}^2.$$

The unit stresses in the steel are less than $15f_c=4500$ lbs/in². Their exact values can be computed from eqs. (7) and (8); the value of k for use in them can be easiest obtained from Plate X.

From this we have $1/k=0.85$; $d/kh=0.765$ and $d'/kh=0.085$. Then from (7)

$$f_s'=15\times297(1-0.085)=4070 \text{ lbs/in}^2,$$
$$f_s=15\times297(1-0.765)=1045 \text{ lbs/in}^2.$$

(2) Change the eccentric distance of the preceding example to 15 inches and solve.

Solution. The eccentricity is $e/h=0.5$, and $M=80,000 \cos 5°\times15=1,195,500$ in-lbs. The beam falls under Case II (see Plate XIII), and for the eccentricity 0.5 and 1% of steel the diagram gives $M/bh^2f_c=0.171$; hence

$$f_c=\frac{1,195,500}{12\times30^2\times0.171}=647 \text{ lbs/in}^2.$$

The intersection of the 1% curve and the 0.5 eccentricity line lies to the left of the curve $f_s/f_c=15$; hence the unit stress in the tensile steel is less than $15\times647=9705$ lbs/in². The exact value can be computed from eq. (13), the value of k for use in it can be obtained easiest from Plate XI.

(3) The breadth of a beam is 12 inches and its height 24 inches. At a certain section the bending moment is 450,000 in-lbs., and the eccentric distance is 4 inches. The working strength of the concrete being 600 lbs/in², how much steel reinforcement, if any, is required?

Solution. The eccentricity is $e/h=\frac{1}{6}$, and hence the beam would be on the border between Case I and II even if no steel were used. With steel, the beam falls under Case I, and

$$\frac{M}{bh^2f_c}=\frac{450,000}{12\times24^2\times600}=0.1085.$$

Entering the diagram, Plate XII, with this value and tracing horizontally to the 0.167 eccentricity vertical, we find their intersection and note that it falls between the 0.6 and 0.8% curves; about 0.7% of steel therefore is required.

(4) In example (3) change the eccentric distance to 12 inches and solve.

Solution. The eccentricity is $e/h = \frac{1}{2}$, and the beam falls under Case II (see Plate XIII). M/bh^2f_c has the same value as in example (3); hence entering the diagram with that value and tracing horizontally to the 0.5 eccentricity vertical, we find their intersection and note that it falls between the 0.2 and 0.3% curves; hence 0.3% is the required amount.

(At first thought it may seem that more steel is necessary in example (4) than in (3) because of the greater eccentricity in the former example, But it should be noted that the thrust N is much less in (4) than in (3). its values being $M/e = 37{,}500$ and 112,500 lbs. respectively.)

TESTS ON BENDING STRENGTH OF BEAMS.

96. Methods of Failure of a Reinforced-concrete Beam.— A reinforced-concrete beam tested to destruction will usually fail in one of three ways:

(*a*) By the yielding of the steel at or near the section of maximum bending moment.

(*b*) By the crushing of the concrete at the same place.

(*c*) By a diagonal tension failure of the concrete at a place where the shear is large.

Methods (*a*) and (*b*) may be called "moment" failures. Method (*c*) is sometimes called a shear failure, but this term is somewhat misleading, as the concrete in such cases does not fail by shearing, but by diagonal tension.

(*a*) As a beam is progressively loaded and the steel has reached its yield point any further load will rapidly increase the deformation. The effect of this is to open up large cracks in the tension side and to raise the neutral axis. This causes a rapid increase in the compressive stress in the concrete and ultimate failure soon occurs by the concrete crushing. Such yielding may also result in final failure by diagonal tension if large shear exists near the place of maximum moment. In either case the primary cause of failure is the yielding of the steel and such failure may properly be called a tension failure. Very rarely can the steel be actually broken in a test. The additional load carried after the yield point is reached depends on the excess strength of the concrete, position of loads, and

other causes, but it is usually not large and cannot be safely considered. The yield point of the steel may therefore be considered its ultimate strength for reinforcing purposes.

(*b*) If the beam is relatively long and the amount of steel is sufficient so that the crushing strength of the concrete is reached before the yield point of the steel, a failure by crushing may result. In this case tension cracks may appear, but will not become large.

(*c*) Diagonal tension failures are likely to occur whenever large shearing stresses exist together with considerable horizontal or moment stresses, and when no special provision is made for such conditions. This kind of failure is fully discussed in Chapter IV.

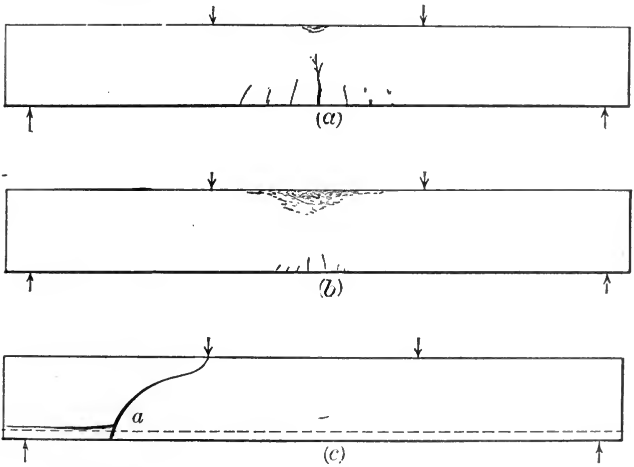


FIG. 29.—Methods of Failure of Beams.

Fig. 29 illustrates roughly the appearance of a beam failing in the different ways.

Final failure often results from stresses which are developed after initial failure has occurred, and while the cause of final failure is important from the standpoint of ultimate strength, yet of more importance in design is the initial failure and its cause. Other conditions besides those already men-

tioned may influence final failure so as often to mislead the observer as to the cause of the initial failure.

97. *Minor Causes of Failure.*—Slipping of the bars may cause failure, but under usual conditions it will not occur; and as it can readily and economically be obviated by proper construction, it need not be considered as limiting the strength of the beam. Failure by the shearing of the concrete near the support is possible where the load is very close thereto, but as the shearing strength of concrete is about one-half the crushing strength, such failures are exceedingly unlikely and need rarely be considered. The usual so-called "shear" failures are in reality diagonal-tension failures. stop

98. Tests of Beams Giving Steel-tension Failures.—In the design of beams it is found advisable to use a proportion of steel such that the ultimate strength of the beam is determined by the strength of the steel in tension rather than by the strength of the concrete in compression. This being the case, it is important to compare results of tests on beams in which failure occurred by yielding of the steel with theoretical calculations of strength.

Fig. 30 represents some results of tests made by the U. S. Bureau of Standards,* which are typical of tests of this class. The beams were 8×10 inches in cross-section and 12 feet long between supports. The concrete was in one case made of a granite aggregate and in the other case of limestone. The yield-point stress of the steel averaged about 41,000 lbs/in^2. Failure occurred in all cases by overstressing of the steel. The line drawn on the diagram is the theoretical strength calculated by eq. (3), Art. 56, using $n=10$ and $f_s=41{,}000$. The maximum compressive in the concrete was about 3000 lbs/in^2, while the ultimate strength in tests cylinders was 3600 lbs/in^2 for the limestone and over 4000 lbs/in^2 for the granite. The values of the initial moduli were about 3,800,000 and 4,400,000, respectively. It would appear, therefore, that a value of $n=10$

* Tech. Papers No. 2, 1912.

for use in the formula would give sufficiently accurate results. A considerable change in n will not affect the results greatly.

It will be noted from Fig. 30 that the test results exceed in all cases somewhat the calculated results, and that there is little difference between the limestone and granite. These results accord very well with the theoretical analysis and indicate that so far as tension failures are concerned the beam will develop the full yield-point strength of the steel. They also

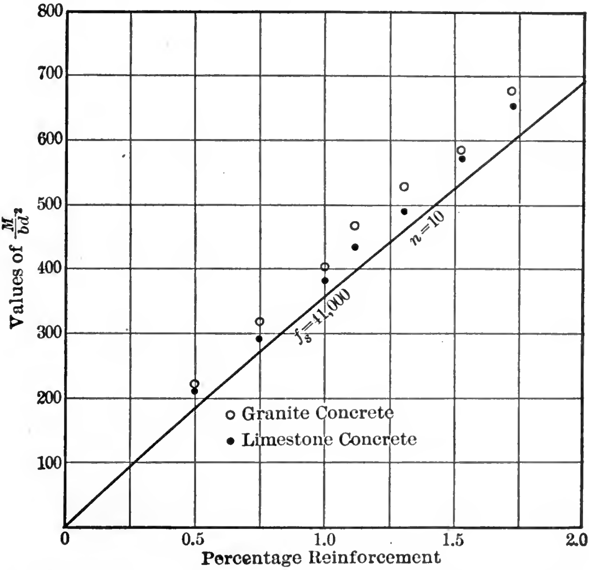


FIG. 30.—Tests of Beams giving Steel-tension Failures.

show clearly that failure will take place at loads but little greater than those corresponding to this yield-point stress. Numerous other tests show similar results.

99. *Results from Individual Tests.*—Numerous tests of beams have been made in which extensometers have been used to measure distortions so that the deformation of the steel and of the extreme fibre of the concrete could be calculated and the neutral axis determined. Results of such measurements of deformations and also of center deflections are shown

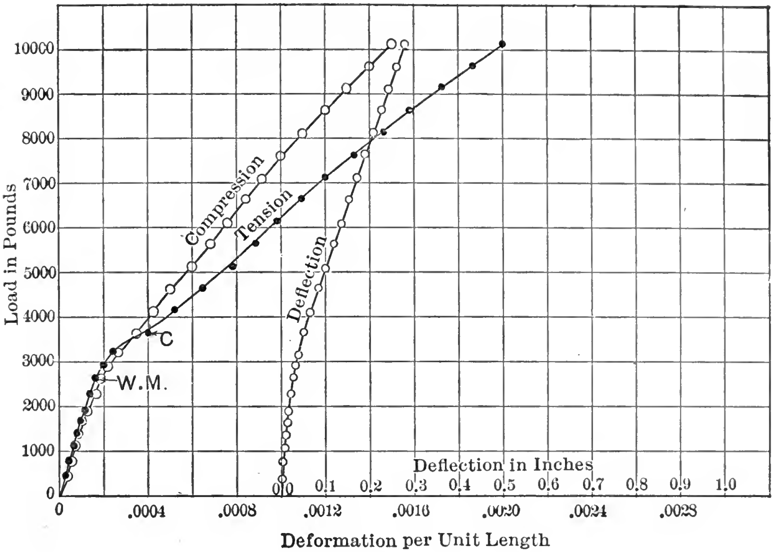
in Figs. 31 and 32 for two typical beams. In Fig. 31 the proportions were such that the failure occurred by diagonal tension; neither the steel nor the concrete was stressed to the limit of failure. During the first stage of the test, up to a load of about 2500 pounds, the deformations in both steel and concrete are proportional to the loads. Up to this point the tension deformation has not been great enough to begin to rupture the concrete, but with increasing loads and deformations the concrete begins to fail, as shown by the appearance of minute cracks shown by faint "water marks" and indicated on the diagram by the letters *W M*. The deformation at the first "water mark" in this case was about .00018, corresponding to a stress of 270 lbs/in^2, assuming a modulus of elasticity of 1,500,000. The first visible crack appeared at the point marked *C*.

The failure of the concrete in tension takes place somewhat gradually and causes a gradual increase in the rate of deformation as indicated by the curved part of the diagram between loads of 2500 and 4000 pounds. After the concrete has ceased to offer any considerable resistance in tension the deformations again become nearly proportional to the loads, but at a different ratio from that obtaining previously, giving nearly straight lines for both steel and concrete—in this case to the end of the test.

In Fig. 32 the amount of steel was small and a tension failure occurred. This is indicated by the great deformations at the end of the test. The curves in the early stages of the test are very similar, in general form, to those in Fig. 31.

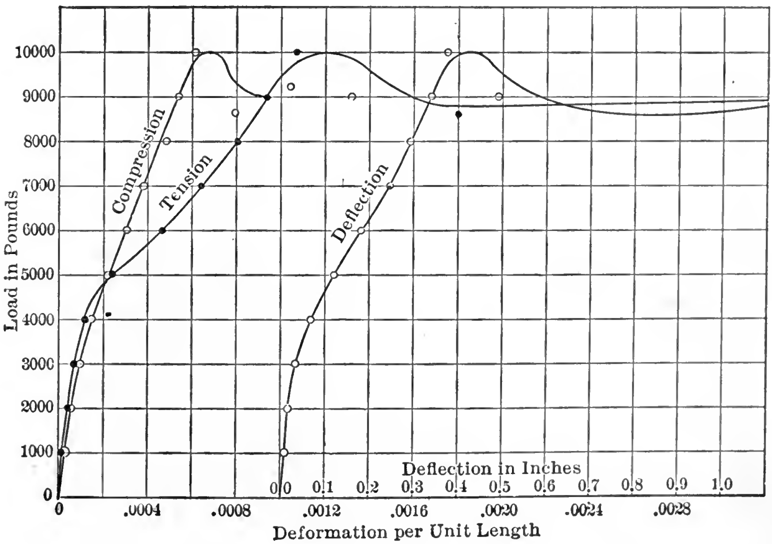
In the case of a compressive failure the curve for compression shows an increased rate of deformation towards the end, somewhat similar to the diagram for simple compression.

100. Position of Neutral Axis and Value of *n*.—Reference to the analysis of Arts. 55 and 56 show that in the calculation of the strength of reinforced beams the determination of the position of the neutral axis is of prime importance. This being known, the strength can be determined with little uncertainty. In determining the position of the neutral axis



Load in Pounds
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
Compression
Tension
Deflection
C
W.M.
Deflection in Inches
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
0 .0004 .0008 .0012 .0016 .0020 .0024 .0028
Deformation per Unit Length

FIG. 31.



Load in Pounds
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
Compression
Tension
Deflection
Deflection in Inches
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
0 .0004 .0008 .0012 .0016 .0020 .0024 .0028
Deformation per Unit Length

FIG. 32.

eq. (1) of Art. 55 shows it to depend only upon the amount of steel used and upon the ratio E_s/E_c or n. The only element of uncertainty is the value of E_c. This is the modulus of elasticity of the concrete in compression and it might be considered sufficient to take the value as determined in the ordinary compression test. However, the variation of E_c for different stresses, and the effect of the tensile stresses in the concrete below the neutral axis (a stress which is properly not allowed for in the resisting moment), make it desirable to compare experimental determinations of the neutral axis with theoretical position for various assumed values of E_c or of n.

Many experiments have been made in which the position of the neutral axis has been determined. Among the best are those by Bach,* made on 1 : 4 gravel concrete, 6 to 7 months old. The beams were 2 m. long and 30 cm. deep and were loaded at quarter points. The observed positions of the neutral axis (values of k), at various loads are given in the following table:

POSITION OF NEUTRAL AXIS.

(VALUES OF k.) (BACH.)

No. of Beams.	Percent Reinf.	Values of k for Various Proportions of Ultimate Load.					Theoretical Values.	
		Initial	$\frac{1}{4}$ Load.	$\frac{1}{2}$ Load.	$\frac{3}{4}$ Load.	Full Load	for $n=12$	for $n=15$
5	0.54	.56	.53	.43	.33	.31	.30	.33
3	0.43	.59	.55	.47	.31	.28	.27	.30
5	1.32	.59	.55	.45	.44	.46	.43	.46

The theoretical positions are also given for $n=12$ and $n=15$. The value of E for this concrete, at a load of 600 lbs/in^2. as determined by compression tests, was 3,300,000 lbs/in^2.

Similar tests have been made on T-beams by Bach and also by Withey.† All of Bach's tests and those on T-beams by Withey are plotted in Fig. 33.‡

* Mit. über Forsch. a. d. Gebiet des Ing., 1907, 45–47.

† Bull. Univ. of Wis., 1908, Vol. 4, No. 2.

‡ See also large number of results in Tech. Papers, Bureau of Standards, No. 2, 1912.

In Fig. 34 are plotted in a different form results of various tests on rectangular beams. On the diagram are also plotted the theoretical positions of the neutral axis for various values of n. The full lines are based on the straight-line stress variation assumption, and the dotted lines on the assumption of a parabolic law in accordance with the method of Art. 68.

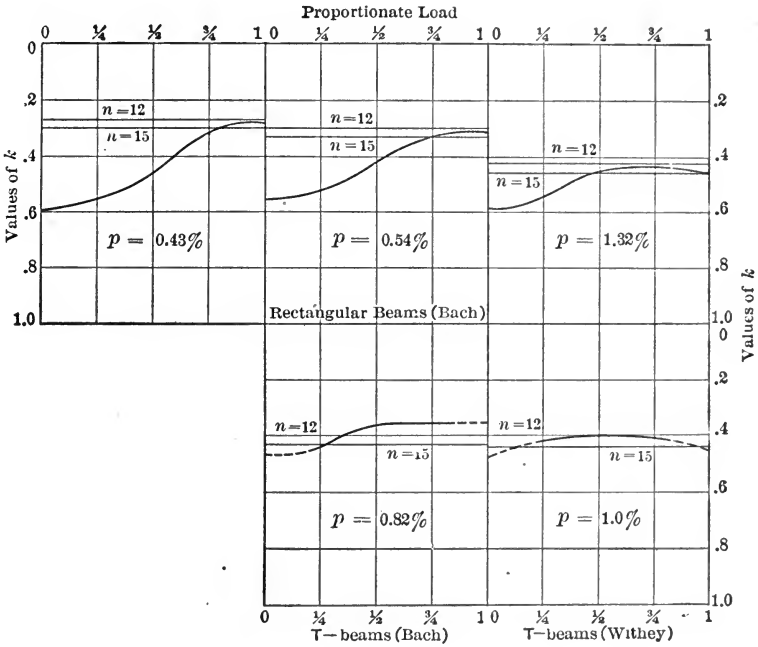


FIG. 33.—Position of Neutral Axis.

The dotted lines have been drawn only for a single value of 15 for n. For a three-quarter load the dotted lines for $n = 15$ would coincide very closely with the full line for $n = 20$. The value of q has been taken at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$, respectively.

It will be noted that for the lower loads and the small percentages of steel the neutral axis is more uncertain and generally lower than for the higher loads and larger percentages. This is due to the relatively large influence of the tensile

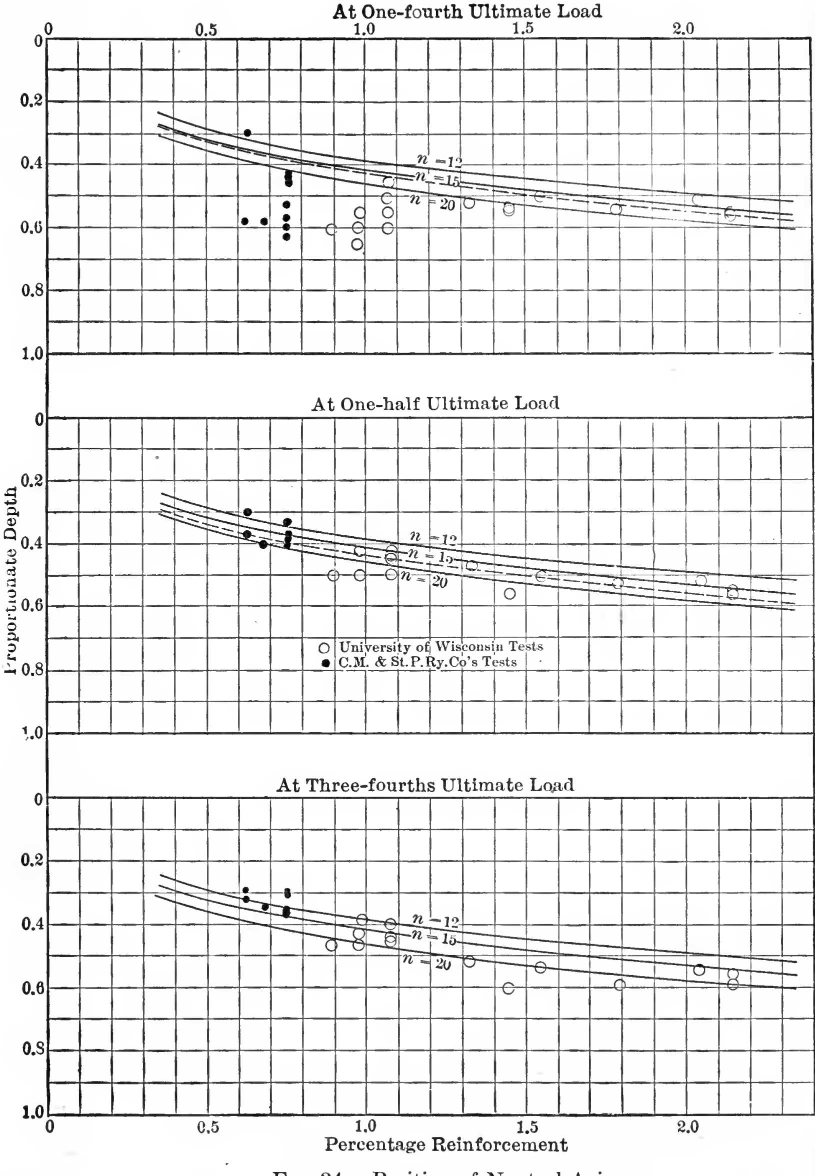


Fig. 34.—Position of Neutral Axis.

strength of the concrete in such cases. The T-beam tests show relatively little of this effect owing to the small area of concrete in tension.

From these results it appears that a value of 15 for n is not too large for calculations of strength of beams under the usual assumptions. This value is the one most generally used, but a value of 12 is also frequently employed. The value of 15 corresponds to a value of E_c of 2,000,000, which is somewhat low as determined by compressive tests. If the comparison between measured and calculated positions of the neutral axis be made on the basis of the parabolic law of stress variation the results will differ considerably in the latter stages of the tests, but very slightly at the quarter load. It should be noted, however, that it is only with the high percentages of steel that the concrete stress reaches nearly to its ultimate value, and hence is the only condition where the full parabolic law can be expected to give consistent and rational results.

In some of the tests whose results are plotted here the concrete was cut away from the steel for the measured distance, leaving it exposed. The position of the neutral axis was very slightly affected.

101. Observed and Calculated Stresses in Steel.—Where the neutral axis is determined by extensometer measurements a check upon theoretical results can be obtained by calculating the stress in the steel in two ways: (1) from the observed deformations at the plane of the steel, and (2) from the known bending moment and known position of the neutral axis. In the first calculation the tensile strength of the concrete, which is neglected, causes some error, especially under light loads, and in the second calculation the exact position of the centroid of pressure in the concrete, especially in the later stages of the test, is to a small degree uncertain, but as the variation in steel stress is only about 2%, using the two extreme assumptions of stress variation, this source of error is not great. Table No. 7 presents several representative results derived from such

calculations. The stresses calculated from moments are based on the assumption that the concrete takes no tension.

Tests have been made at the University of Illinois and at the University of Wisconsin in which the rods have been exposed for a considerable distance along the center of the beam, and thus have been much less affected by any possible tensile stress in the concrete. Measurements of extension made in such cases show little variation from those made on the ordinary beam.

Table No. 7.

STRESSES IN STEEL REINFORCEMENT.

Authority	Per Cent Reinforcement.	Observed Position of Neutral Axis, k.	Calculated Stress in Steel. lbs/in^2. From Moments.	Calculated Stress in Steel. lbs/in^2. From Extensions in Steel.
Talbot; *Bull. Univ. of Ill., 1906.*	.74	.410	33,100	36,000
	1.23	.470	35,000	36,000
	1.60	.501	29,500	35,400
	1.66	.505	30,600	30,000
	1.84	.606	25,600	27,200
	1.84	.552	28,300	30,000
Withey; *Bull. Univ. of Wis., 1907.*	2.9	.670	3[illegible],200	36,000
	2.9	.6[illegible]0	31,600	33,000

Considering the nature of such experiments the results obtained may be considered as according with theory very satisfactorily.

102. Compressive Stresses in Concrete in Beams and in Compression Specimens.—An important question relating to proper working stresses is whether the ultimate compressive strength of concrete in a beam is the same as determined by a direct compression test.

The results of certain tests indicate that the compressive strength and ultimate deformation in a beam may be somewhat greater than in a prismatic compressive piece; and it would seem that the differences in condition are sufficient to

make such a difference possible. In a compression specimen the material is free to shear in any direction, thus limiting the strength of the specimen to its weakest shearing plane. In a beam the (shear) failure is practically confined to planes perpendicular to the side of the beam. Furthermore, in a beam the material is not subjected to the secondary stresses due to possible poor bedding of the test specimen or non-parallel motion of the testing machine, as is the case in compression tests.

In most of the tests reported both the beams and the accompanying compression specimens have been hardened in air. Under these conditions there is usually some drying-out effect resulting in a weaker concrete than if hardened in water, and owing to the smaller dimensions of the compressive specimens the effect will be greater with them than with the relatively large beams. Many tests have therefore shown a compressive strength of concrete in the beam considerably greater than results obtained on cubes. When both beam and cube are hardened in water the results do not differ greatly. The following are some results obtained on tests made relative to this point.* The beams were $5\frac{3}{4}'' \times 6''$ net section and 5 ft. span. They were reinforced with $2\frac{1}{2}\%$ of steel and gave compression failures. The cubes were 4 inches in dimension and the cylinders 6 inches in diameter by 18 inches high.

		Stress in Concrete at Rupture, lbs/in².		
		Beam.	Cube.	Cylinder.
Hardened in air	1..........	1770	1187	1380
	2..........	1460	1350	1295
Hardened in water	3..........	1810	1450	1265
	4..........	1850	1750	1680

The stresses in the beams were calculated on the basis of the

* Bulletin No. 1, Vol. 4, Engineering Series, University of Wisconsin, 1907.

parabolic variation of stress, the neutral axis being determined by extensometers.

It will be seen that in case of the specimens hardened in air there is a marked difference in strength, but where hardened in water the difference is much less. The difference is hardly sufficient to warrant much consideration in the determination of working stresses.

103. Conclusions.—The comparison of experimental results with theoretical analysis herein given shows that the simple beam theory as generally employed, neglecting the tension in the concrete, can be used with confidence. In particular, the results appear to show that, calculated on the basis of such theory, the yield point (commonly called the elastic limit) of the steel may safely be taken as its ultimate strength in reinforced beams; that the crushing strength of concrete as determined by tests on cubes hardened under exactly similar conditions as the beams will be fully realized in the beam; that for working loads the straight-line law of stress variation is sufficiently exact; that the value of n may be taken at about 15, but that great accuracy in this respect is unnecessary; that for ultimate values, especially where the concrete is near failure, the parabolic assumption of stress variation may well be used.

CHAPTER IV.

SHEAR AND BOND STRESS.

THEORY AND GENERAL RELATIONS.

104. General Relations.—In Art. 45 it was shown that the direction and intensity of the maximum tensile stress at any point in the body of a beam are dependent upon both the shearing and the bending stress existing at the point in question, and that where the bending stress is small the shearing-stress is the chief factor. The general formula for maximum tensile stress is here repeated. It is

$$t=\tfrac{1}{2}f+\sqrt{\tfrac{1}{4}f^2+v^2}, \quad \ldots \ldots \quad (1)$$

and its direction is given by the equation

$$\tan 2\theta=2v/f, \quad \ldots \ldots \quad (2)$$

in which f=horizontal fibre stress (due to bending) at the point, v=vertical or horizontal shearing-stress, t=maximum tensile stress and θ=inclination of the maximum tension to the horizontal. At the neutral plane, for example, the maximum tensile stress is at an inclination of 45 degrees and is equal in intensity to the shearing-stress at that point. Near the end of a simple beam where the bending moment is small, the value of f in eq. (1) is small and the tensile stress t is nearly equal to v at all points in a section.

In order, therefore, to be able to investigate fully the inclined tensile stresses in a beam and to provide proper reinforcement at all points, it is necessary to determine the distribution of the shearing-stresses throughout the depth of the beam.

The stress existing between reinforcing steel and concrete tending to cause slip, or the bond stress, is also dependent

upon the shear, it is similar to the action between the flange and web of a girder caused by the horizontal shear. The bond stress is, therefore, conveniently discussed in connection with shear.

105. Shearing-Stresses in Reinforced Beams.—In Art. 45 the variation in shearing-stress in a homogeneous beam was discussed and the general formula given for the intensity of shear at any point (see eq. (1)). In a reinforced beam the variation in shear differs from that in the homogeneous beam, owing to the concentration of tensile stress in the steel.

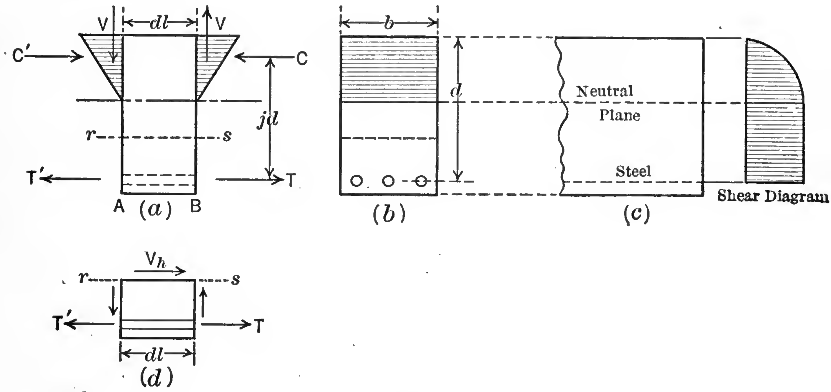


FIG. 1.

106. *Rectangular Beams.*—In Fig. 1, is represented a short portion AB of beam length dl, with all shearing and bending stresses indicated. The tension in the concrete is neglected. The total vertical shear on each end is V, the increment of load applied between A and B being neglected. Considering any horizontal section, r–s, below the neutral axis, Fig. (d), the total horizontal shear V_h on this section is equal to $T'-T$, and the intensity of the horizontal shearing-stress will be

$$v=\frac{V_h}{bdl}=\frac{T'-T}{bdl}. \qquad (3)$$

The increment of tensile stress, $T'-T$, may conveniently be replaced by a function of V, from the moment equation,

$(T'-T)jd=Vdl$, whence $T'-T=\frac{Vdl}{jd}$. Substituting in (3), we have the more convenient expression for intensity of shear,

$$v=\frac{V}{bjd}. \quad \ldots \ldots \ldots \quad (4)$$

Eq. (4) gives the intensity of the horizontal shearing-stress on any plane between the neutral axis and the steel. It is to be noted, also, from the general principles given in Art. 45, that the vertical shearing-stress per unit area at any point is equal to the horizontal. Above the neutral axis the shear decreases according to the parabolic law as in a homogeneous rectangular beam. Fig. (*c*) represents the law of variation for the case under discussion.

Using $\frac{7}{8}$ for an approximate value of j (see Art. 56), we have approximately

$$v=\frac{8}{7}\frac{V}{bd}, \quad \ldots \ldots \ldots \quad (5)$$

that is, the shearing-stress at the neutral axis (equal to the maximum) is about one-seventh or 14% more than the average value obtained by dividing the total vertical shear by the sectional area.

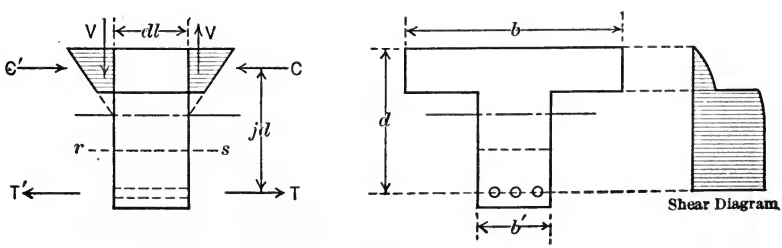


Fig. 2.

107. *T-beams.*—Applying the same method of analysis as in the previous article, it is obvious that the shearing-stress on any section r–s (Fig. 2), in the stem below the neutral axis is also given by eq. (4), substituting b' for b. The value of jd

is the lever arm of the stress couple C–T. Hence, for T-beams,

$$v=\frac{V}{bjd}. \qquad (6)$$

The shearing-stresses in a T-beam below the flange are therefore practically the same as in a rectangular beam having the same depth and the same width as the *stem* of the T. The slab aids in reducing the shear only by its effect in increasing slightly the value of j.

108. *Beams Reinforced for Compression.*—In beams reinforced for compression, eq. (4) will still apply, the value of jd being the distance between the tensile steel and the resultant of the compressive stresses as shown in Art. 85.

109. *The Average Shearing-Stress.*—Since the value of j varies only within narrow limits, it is quite as satisfactory for comparative purposes to use the average value of the shearing-stress,

$$v'=\frac{V}{bd}, \qquad (7)$$

in which b is the breadth and d is the net depth of the beam. In T-beams, b is the breadth of the stem and d is the total depth from top of beam to steel. The true maximum shear will generally be from 10 to 15% higher than the average value thus determined. Eq. (7) may also be used for purposes of design if the working stress is selected accordingly.

110. **Formulas for Bond Stress.**—The stress on the bond between steel and concrete (Fig. 1, Art. 106), will be equal to $T'-T$ on the length dl.

If U denotes the bond stress per lineal inch, we then have

$$U=\frac{T'-T}{dl},$$

whence we derive

$$U=\frac{V}{jd}. \qquad (1)$$

The bond stress per unit area will be equal to U divided by the sum of the perimeters of the steel sections. Or, if o = perimeter of one bar, Σo = sum of perimeters, and u = bond stress per unit area, we have

$$u=\frac{V}{\Sigma o \cdot jd}. \qquad (2)$$

The bond stress, therefore, varies directly with the total shear V and inversely with the perimeters of the rods.

111. *Bond Stress for Compressive Reinforcement.*—The question of bond stress for compressive reinforcement will seldom come into consideration. If required it can be calculated most readily by comparing it with the bond stress in the tensile steel. Whatever may be the amounts and positions of the compressive and tensile steels, the total stresses in the two sets of bars will be proportional to their areas and distances from the neutral axis. Using the notation in Art. 84 we have, therefore, $\frac{C'}{T}=\frac{A'(kd-d')}{A(d-kd)}$. This being true at any section, it follows that the *increment* of stress per lineal inch in the reinforcement (the bond stress) will also be proportional to the same quantities. Hence, if U' = bond stress per lineal inch along the compressive reinforcement, we have

$$\frac{U'}{U}=\frac{A'(kd-d')}{A(d-kd)}, \qquad (3)$$

hence, from (1),

$$U'=\frac{V}{jd}\times\frac{A'(kd-d')}{A(d-kd)}. \qquad (4)$$

Since the compressive steel will generally be nearer the neutral axis than the tensile steel it follows that if the compression bars are no larger in diameter than the tension bars, the bond stress in the former will be no greater than in the latter.

112. Diagonal Tension and Shear.—It has been shown in Art. 45 (Fig. 4) that the direction of the maximum tensile

stress at any point in the interior of a homogeneous beam is, in general, not horizontal but is inclined at some angle to the horizontal. The direction and intensity of this tensile stress are functions of the horizontal fibre stress and of the shearing-stress at the point in question. At the bottom fibre the maximum tension is horizontal; at the neutral axis it is at 45 degrees inclination, and equal in value to the shearing-stress. At sections of zero shear it is horizontal at all points.

It will be seen from a consideration of eq. (5), Art. 45, that where the bending moment is large and the shear is small, (sections near the center of a simple beam) the influence of the shearing-stress upon the direction and amount of the maximum tension is small; the maximum values are but little greater than the horizontal fibre stresses. But at sections of large shear and small moment the effect of the shear is great and the inclined tensile stresses near the neutral axis may be larger than the horizontal stress at the bottom.

In the reinforced beam the same relation exists between shear, direct stress and inclined tension at any given point as given by eq. (5) as this equation is perfectly general in scope. In the reinforced beam, however, the direct stress, f, does not vary in the same manner as in the homogeneous beam on account of the concentration of the tension in the steel; so that, as a result, the direction of maximum tension at various depths is somewhat different from that shown in Fig. 4. In this case large shearing-stresses exist immediately above the steel, hence the maximum tensile stresses become considerably inclined just above the steel, the exact direction depending upon the relation between the shear and the horizontal tension. The problem is complicated by the fact that in the region of large moment the stress and deformation in the steel is so large that, under working loads, the concrete is always cracked more or less near the bottom, and at such points the concrete possesses neither tensile nor shearing strength.

It will be of assistance in gaining a knowledge of this question to consider in some detail the development of stresses

in a reinforced concrete beam as it is progressively loaded, and in particular the tendency for the concrete to crack and the direction of such cracks. Consider a uniformly loaded beam, Fig. 3, and assume the load to be progressively applied. Assume, further, that the reinforcement is uniform throughout,

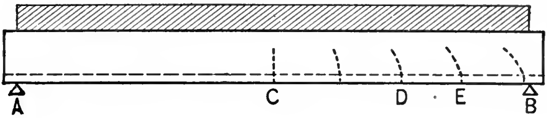


FIG. 3.

consisting of horizontal bars only. Fig. 4 shows the form of moment and shear curves at all times, the actual values depending, of course, upon the load. As the load increases, the tensile stress in the steel rods will increase, and when this

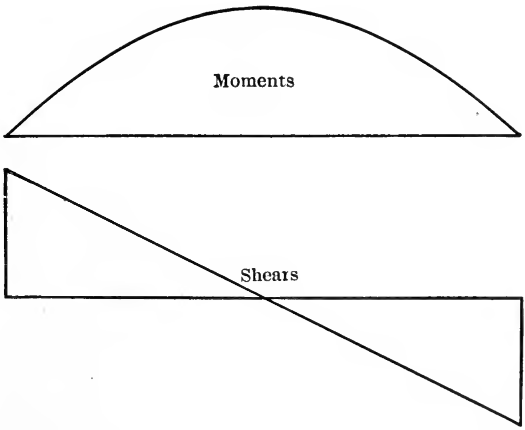


FIG. 4.

stress has reached a certain value (about 4000–6000 lbs/in²), the concrete will begin to crack (see Art. 41). This will occur first at and near the center. As the load increases, these first cracks will gradually extend towards the neutral axis and cracking will begin at points D, E, etc., farther and

farther from the centre as the stresses in steel and concrete increase. At and near the centre where the shear is zero, or very small, the cracks will be vertical, as the direction of maximum tension is horizontal. As we pass towards the support the shearing-stresses become larger, so that the direction of maximum tension just above the rods becomes more and more inclined to the horizontal, and the cracks will not be vertical but will take an inclined direction, the inclination being greater as the end is approached. At point *E* the crack will be more inclined than at *D*, as the relative shearing-stress will be greater; near the support where the moment stresses are nearly zero, the theoretical direction of maximum tension will be at 45 degrees at all depths. The form of the cracks is generally curved, somewhat as shown, corresponding roughly to the change in direction of maximum tension. The general direction of cracks under progressive loading is well shown in Fig. 12, Art. 127.

113. Failure from Diagonal Tension.—Consider a beam with horizontal reinforcement only. Under ordinary working loads the reinforcement at the center is calculated for a unit stress of about 16,000 lbs/in². It must happen, therefore, that under such loads the concrete is actually cracked in the vicinity of maximum moment. As shown in the preceding article, these cracks will be vertical, or nearly so, and hence at right angles to the reinforcement. So long as this condition exists no danger is involved, as the opening of the cracks is strictly limited by the deformation of the steel. Where the cracks are inclined, however, the reinforcement is not at right angles to the cracks and a different action is involved. Suppose, for example, that near the end of the beam, Fig. 5, a crack *ab* starts at about 45 degrees inclination. As the load is increased it tends to open up as shown in Fig. (*b*). The movement of the point *c* with reference to *d* is at 45 degrees inclination, that is to say, the point *c* tends to move downward as well as towards the left with respect to *d*. The horizontal rod is effective reinforcement against excessive horizontal movement, but it offers

very little resistance against vertical motion. A very small opening of the inclined crack thus brings a heavy *vertical* load on the bars at *c*, which is transferred across the crack by the bars and causes heavy tensile stress along the line *de*. A horizontal crack then starts at *d* and the concrete very quickly strips or tears off along the line *de* and the beam fails.

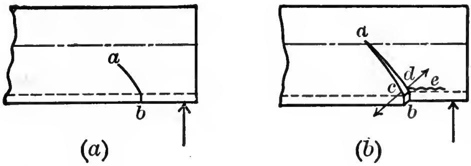


FIG. 5.

It is thus seen that with horizontal rods only, inclined cracks caused by diagonal tension are dangerous, as the horizontal rods are not in a favorable position to prevent extension of such cracks and failure of the beam. That is to say, *horizontal rods only are not adequate reinforcement against diagonal tension.*

It is obvious that the adequacy of horizontal rods depends largely upon the inclination of a diagonal crack. Where such inclination is slight, as at points where the shear is small and moment large, horizontal rods may be sufficient; it is in regions of large shear where they are likely to be inadequate. It is also to be noted that the actual unit stress in the horizontal rods is of importance, a low unit stress tending to keep the deformations and cracks in the concrete small, thus limiting also the vertical movements at diagonal cracks.

114. The Shear as a Measure of the Diagonal Tension.—It has been shown in the preceding article that tension cracks are not likely to form at a dangerous inclination except where the shear is large, and in the case of simple beams, near the end where the moment stresses are small. The strength of a beam in diagonal tension is not a simple function of the shear, but as shown in Art. 112 it depends also upon the horizontal

tension or the bending-moment stresses in the concrete. These will, in turn, depend upon the actual bending moment at the section of failure and the amount of horizontal reinforcement, a large percentage of reinforcement reducing the horizontal deformation and therefore the tension in the concrete, and tending to strengthen the beam as regards failure in diagonal tension. The strength of the beam therefore depends upon the relation between shear and bending moment and upon the amount of reinforcement. The chief factor is, however, the shearing stress. Roughly speaking, therefore, it may be said that the diagonal tension of importance in this sense may be measured by the shear. At the neutral axis this diagonal tension is always a maximum at 45 degrees inclination and is equal to the shearing-stress; and while the diagonal stress at points below the neutral axis may be somewhat greater, the results of tests show that the diagonal tensile stress at the neutral axis, equal to the shearing-stress $\left(v=\frac{V}{bjd}\right)$, is a fair measure of the stresses which need to be provided for. It is found, for example, that when the shearing stress in the lower part of the beam as given by the formula $v=V/bjd$ reaches a value of 125 to 150 lbs/in^2, diagonal cracks are likely to appear, and if the reinforcement consists of straight rods only, failure is likely to follow very quickly in the manner shown in Fig. 5. The strength of a beam in diagonal tension can, therefore, be very approximately determined by a calculation of the maximum shearing stresses.

115. Methods of Reinforcement Against Diagonal Tension.—There are in use many methods of placing steel in a beam so as to reinforce it against diagonal tension failure. Theoretically, the most effective way to reinforce against tension failure in any direction is to place reinforcement across the probable lines of rupture, or in the direction of the maximum tensile stresses. From these considerations the ideal web reinforcement would be a system of rods arranged somewhat as shown in Fig. 6 attached at their lower ends to the horizontal rods,

or consisting of numerous horizontal rods bent up as indicated. The figure also indicates roughly the manner in which the inclination of diagonal cracks near the bottom tends to vary from nearly vertical at the centre to a large inclination at the end. The exact conditions depend upon the nature of the loading, concentrated loads tending to extend the region of large shear to greater distances from the support. It is, however, not practicable nor necessary to have the inclination of the reinforcing rods exactly the same as the lines of maximum

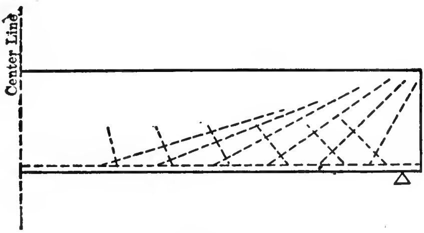


Fig. 6.

tension, and various arrangements will serve to accomplish the purpose.

The most commonly used methods of arranging shear reinforcement are: (1) bent up bars, (2) vertical secondary members called stirrups, and (3) inclined secondary members. Fig. 7 illustrates these various arrangements.

Fig. (*c*) illustrates a method adapted to relatively small shearing-stresses. A part of the horizontal rods are bent up at a small angle. For heavier stresses several rods may be bent up as in Fig. (*f*). Fig. (*d*) shows the use of the vertical stirrup. This device has long been successfully used and its design has been well standardized. Figs. (*e*) and (*g*) show a combination of bent rods and stirrups which is a very effective arrangement. Fig. (*h*) shows inclined stirrups or secondary members connected to the horizontal bars. A special form of bar designed to give the same result is shown in Fig. 9 Art. 35.

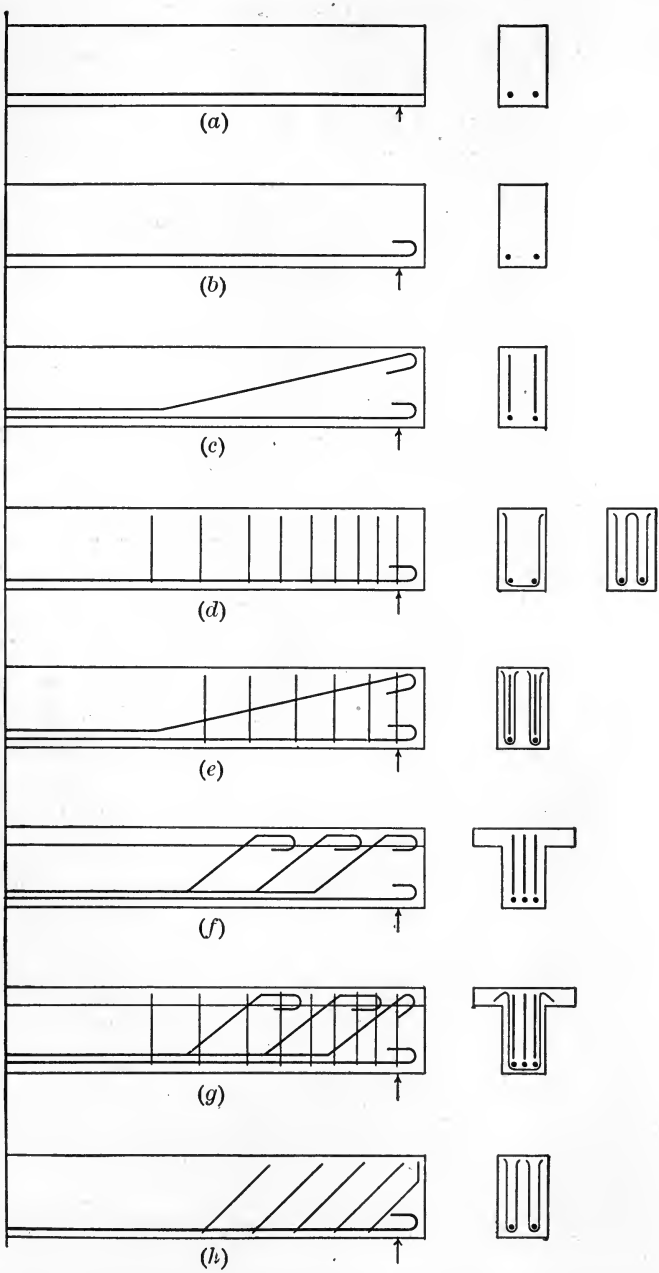


Fig. 7.—General Methods of Arranging Beam Reinforcement.

116. Action of Shear Reinforcement.—To aid in understanding the action of steel placed in various ways, consider the typical diagonal tension failure, Fig. 8, as it occurs where only horizontal rods are used. The inclined crack at *a* usually appears first, due to rupture of the concrete in tension. To assist in preventing this rupture in its initial stage the most efficient reinforcement would be such as supplied by the inclined rod 1, fastened to the horizontal bar, or by the bent end of one of the horizontal bars. Such inclined reinforcement is in the most favorable position to resist further deformation; in fact, the deformation in the concrete previous to rupture, produced by the tensile stresses, causes some elongation of the reinforcement, thus developing some resistance in the reinforcement before cracking takes place. A vertical rod 2, can hardly be as effective as the inclined rod in preventing initial rupture, for so long as the concrete is intact the deformation along a vertical line is practically zero, owing to the combined action of web tension and web compression at right angles to each other. However, the deformation which takes place in concrete before rupture in tension is so small that there is not much difference in the behavior of beams, whether reinforced by inclined or by vertical reinforcement, provided the reinforcement be adequate in amount and properly arranged. Compare beams *C* and *D* of Fig. 12, Art. 127.

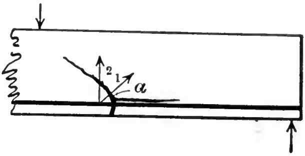


Fig. 8.

Vertical stirrups should be placed a distance apart not greater than about three-fourths the depth of the beam. They should be looped around the horizontal *tension* reinforcement, so as to be firmly anchored thereto. In continuous beams they must be anchored to the *upper* reinforcement at all points of negative moment. This requirement is often overlooked. The bond strength of stirrups must also be carefully provided for. Assuming that an inclined crack, Fig. 8, may extend as far as the neutral axis, it follows that the full strength of bond

for the stirrups or other web reinforcement should be provided in the portion of the beam *above* the neutral axis (or in general in the compression area). Inclined reinforcement must be securely attached to the horizontal rods. If so attached it is very effective, not only with respect to shear, but also in increasing the bond strength of the main bars. It should be spaced a distance apart not greater than the depth of the beam.

Bent rods alone are apt to be of limited value, owing to the difficulty of providing rods close enough together. Convenience of horizontal reinforcement calls for comparatively few rods of large size, which provides too few for effective diagonal reinforcement. Where large rods are bent up, the length of the bent end should be made sufficient to develop the requisite bond strength. The use of hooked ends as shown in Fig. 7 is good practice.

In the case of continuous girders it is convenient to extend the bent rod horizontally at the top over the support to furnish a part of the tension reinforcement. A very satisfactory arrangement of shear reinforcement is a combination of bent rods and vertical stirrups, and especially is this the case in continuous beam construction. Further details are given in Chapter V.

TESTS ON BOND STRENGTH.

117. Nature of Bond Resistance.—When slipping takes place between a plain, smooth bar and the surrounding concrete under a gradually increasing force, the progressive action appears to be as follows: Until the load reaches a value sufficient to produce a bond stress of 200 to 300 lbs/in² (depending upon conditions) there is no measureable movement; the adhesion between cement and steel appears to hold the two materials firmly together. As the load increases, slipping begins, bringing into action the frictional resistance. This resistance increases as the slip increases and reaches a maximum for a slip of about .01 inch. After this point the resist-

ance gradually falls off. The resistance to slip depends upon the smoothness of the rods, and the character and age of concrete. In the case of bars having corrugations or ribs, the initial action is about the same as for plain bars, but the resistance continues to increase with increase of slip until failure occurs by the splitting of the concrete or the shearing through of the rods at a relatively high value.

118. Methods of Testing.—Tests of bond are generally made by imbedding the rod in a block of concrete and pulling it therefrom, the rod being stressed in tension and the concrete in compression. Tests have also been made by *pushing* the rods through the block. Neither of these methods of testing is entirely satisfactory for use in beam analysis, as they do not altogether simulate the action in a beam where both the rod and the concrete are in tension. Various experimenters have, accordingly, made efforts to determine the bond strength by tests on beams.

119. Results of Tests in Direct Tension.—Table No. 8 contains in condensed form the results of some of the most important tests made by direct tension. Additional tests by direct tension are given in Table No. 9.

The variation of bond resistance during a test is illustrated in Fig. 9, the solid line being the average load-slip curve for the third group of tests of Table 8. It is seen that the maximum resistance occurs at a slip of about .01 inch, after which the resistance gradually falls off. Generally the slip begins at a load of from 60 to 80% of the maximum. After a slip of .10 inch occurs, the frictional resistance is still about $\frac{2}{3}$ of the maximum resistance. In these tests the slip was measured at the free end.

In general, the stronger the concrete the greater the bond strength, whether the difference is due to age or richness of mixture, the bond strength being approximately proportional to the strength of the concrete. The results of Abram's tests show a value for the maximum bond resistance of about $\frac{1}{4}$ the compressive strength of 6-inch cubes, and the resistance

TABLE No. 8.

BOND TEST BY DIRECT TENSION, PLAIN BARS.

All Concrete 1 : 2 : 4; 60–90 Days Old.

Authority.	Steel Bars.		Depth Embedded, Inches.	Bond Resistance.		
	Kind.	Size, Inches.		At End Slip of .0005 In.	At End Slip of .001 In.	Maximum.
(1)	Round	$\frac{3}{16}$ to $\frac{3}{4}$	6	....	....	400
		$\frac{3}{16}$ to $\frac{3}{4}$	8	....	....	310
(2)	Round	$\frac{1}{2}$ to $1\frac{1}{4}$	25 diam.	....	....	410
		$\frac{1}{2}$ to $1\frac{1}{4}$	40 diam.	....	....	390
(3)	Round	$\frac{1}{2}$	8	323	339	381
		$\frac{5}{8}$	8	266	295	405
		$\frac{3}{4}$	8	275	303	387
		1	8	247	281	385
		$1\frac{1}{4}$	8	269	296	397
(3)	Flat	$\frac{1}{2}\times 1$	6	359	395	459
		$\frac{1}{4}\times 2$	4	239	263	293
(3)	Polished Round	1	5	149	....	152
		$\frac{3}{4}$	5	137	146	160
		$\frac{3}{4}$	6	170	192	255

(1) Withey; *Bull., Univ. of Wis.*, No. 175, 1907.
(2) Van Ornum; *Eng. News*, Vol. LIX, 1908, p. 142.
(3) Abrams; *Bull. No.* 71, *Univ. of Ill. Eng. Exp. Sta.*, 1913.

at a slip of .0005 inch about $\frac{1}{6}$ of this compressive strength. In terms of the strength of cylindrical specimens, these proportions are estimated to be about 19% and 13% respectively.* Tests indicate little difference due to size of rod. The bond strength of flat bars is considerably less than round bars, and that of polished or smooth bars very much less than ordinary bars. Rusted bars show considerably higher bond resistance than bars with the ordinary mill scale, such as usually tested.

120. Bond Strength of Deformed Bars.—The initial action of deformed bars is very similar to that of plain bars. When the adhesion is broken a small slip takes place under increasing load at about the same rate at first as for plain bars. After

* See Bull. No. 71, Univ. of Ill., 1913.

reaching a movement of about .01 inch, however, the resistance continues to increase as the projections begin to bear hard against the concrete. If splitting of the concrete does not take place, the resistance continues to rise to a high value, but conditions are not usually such as to make available this higher value of the bond resistance. Either the concrete splits or the slip is so great as to cause failure of the beam in other ways.

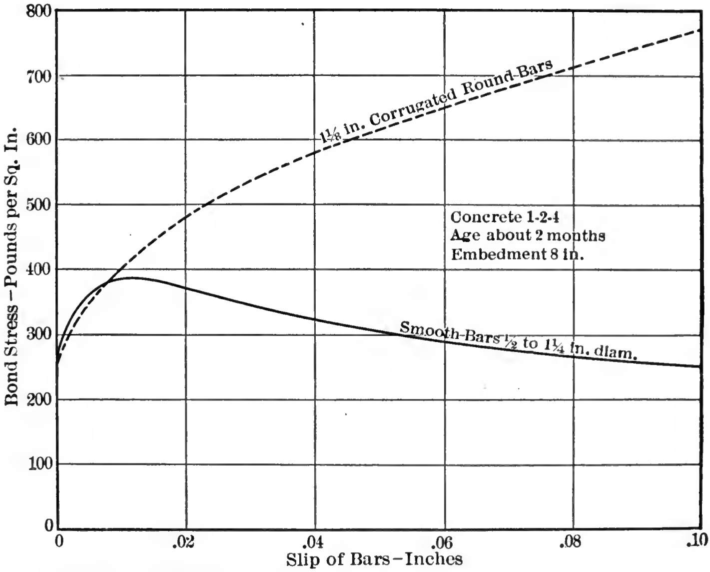


Fig. 9.—Load-slip Curves for Plain and Corrugated Bars.

Important tests on deformed bars are reported in the bulletin of Mr. Abrams' already referred to. Fig. 9 shows typical curves of load-slip relations of a corrugated round bar as compared to plain round. It is to be noted that the action is practically the same up to a slip of .01 inch, beyond which the resistance of the deformed bar continues to increase, but at a decreasing rate. Table 9 gives average results of pull-out tests on several types of deformed bars.

It is to be noted that for a slip of .01 inch, the deformed

Table No. 9.

BOND TESTS BY DIRECT TENSION, DEFORMED BARS.

Tests by D. A. Abrams.*

All Concrete 1 : 2 : 4; 60–80 days old.

Size and Kind of Bar.	Bond Stress at End Slip of (Ins.) .0005	.01	.05	Per Cent of Bond Strength of 1–1¼-in. Plain Round Bars at Slip of .01 In.
1-in. and 1¼-in. plain round (average)	270	382	310	100
¾-in. cup	321	468	935	116
1-in. cup	290	504	916	124
¾-in. lug (straight)	306	454	647	112
1-in. lug (straight)	324	490	562	121
¾-in. lug (twisted)	239	369	684	91
1-in. lug (twisted	251	403	556	99
¾-in. corrugated square (*A*)	245	469	820	116
1-in. corrugated square (*A*)	334	588	945	145
¾-in. corrugated square (*B*)	279	462	692	114
1-in. corrugated square (*B*)	247	497	741	123
$\frac{9}{16}$-in. corrugated round (*C*)	336	526	876	130
1⅛-in. corrugated round (*C*)	236	334	624	82
¾-in. Thacher bar	248	415	564	102
½-in. twisted square bar	324	371	334	92
1-in. twisted square bar	249	343	353	85
1-in. threaded round	545	679	734	167

* *Bull. Univ. of Ill. Eng. Exp. Sta.*, No. 71, 1913, p. 68.

bar offers about the same resistance as a plain round bar. The round bar with the screw threads showed much higher resistance to early slip than any of the others. All except the twisted square bars showed largely increased resistances at a slip of .05 inch, much higher than plain bars, but such large values of slip and resistance are hardly available in design except as tending to retard the ultimate failure and collapse of the structure. The square twisted bars showed smaller resistance at .01 inch slip than the plain bars, but the resistance was better maintained for continued movement.

Results similar to the above values were obtained by Mr. T. L. Condron.* He found a maximum resistance for plain bars at a slip of about .01 inch. For twisted bars the resistance continued to increase slightly under movements up to $\frac{1}{16}$ inch

* Jour. West. Soc. Engrs., 1907, Vol. XII, p. 100.

TABLE No. 10.

BOND TESTS ON BEAMS.

UNIVERSITY OF WISCONSIN, 1907.

Concrete 1 : 2 : 4; age 60 days.

No. of Tests.	Diam. of Rod, Inches.	Bond Strength.	Average of Group (A).	Average Bond Strength by Direct Tension (B).	Ratio B : A.
38		345			
39	$\frac{3}{8}$	298	278	394	1.42
40		190			
41		361			
42	$\frac{1}{2}$	312	286	455	1.54
43		186			
7		362			
8		264			
9	$\frac{5}{8}$	201	272	467	1.75
36		254			
37		278			
44		207			
45	$\frac{3}{4}$	289	264	502	1.90
46		295			
47		136			
48	1	174	163	487	2.99
49		180			

or more, while indented bars (the Thatcher and corrugated bars) showed steadily increasing resistance under increased slip up to rupture. The actual bond stress for .01 inch of movement was 400–600 lbs/in² for the Thatcher and the corrugated bars, 250–400 lbs/in² for the twisted bars, and 175–300 lbs/in² for the smooth bars.

121. Results of Bond Tests on Beams.—In a series of tests by Mr. Withey* at the University of Wisconsin, test beams were arranged as shown in Fig. 10. The stresses in the exposed rods were determined by means of extensometers. The conditions were similar in many respects to those obtaining in an ordinary beam, but the beam was prevented from failing by the upper auxiliary rods. Table No. 10 gives the principal results of these tests. The table also contains results of comparative tests made at the same time by the usual direct tension method. The last column gives the ratio of the two results.

* Engineering Record, 1908, LVII, p. 798.

The rods were of ordinary mild steel and were free from rust. The beams were 5×5 inches in section and 5 feet 6 inches long.

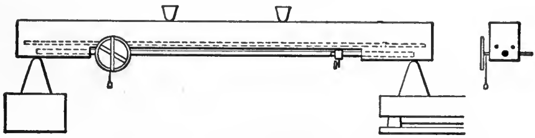


FIG. 10.

These tests indicate that the bond was not affected by size of rod except in the case of the 1-inch size. This difference is undoubtedly due to other factors not explained. Excepting the tests on this size the results are quite uniform, the average of all being 275 lbs/in², with maximum variations of 32% below and 32% above the average. The results obtained by direct tension are much higher, averaging about 475 lbs/in², or 75% greater.

Table No. 11 gives results of tests on beams by Mr. Abrams, in which beams 8×12 inches in section and 5 to 10 feet long were used, reinforced by a single bar. Loads were applied at the third points and the bond stresses in the rods were calculated by the formula $u = V/\Sigma o \cdot jd$. Slip of rods was carefully measured at the ends of the beams.

TABLE No. 11.

BOND TESTS ON BEAMS.

UNIVERSITY OF ILLINOIS, 1913.

Concrete 1 : 2 : 4; age 2–8 months.

Kind of Bars.	No. of Tests.	Bond Stress.		
		At First End Slip.	At End Slip of .001 In.	Maximum Value.
1- and $1\frac{1}{4}$-in. plain round	28	245	340	375
$\frac{3}{4}$-in. plain round.......	3	186	242	274
$\frac{5}{8}$-in. plain round.......	3	172	235	255
1-in. plain square.......	6	190	248	278
1-in. twisted square.....	3	222	289	337
$1\frac{1}{8}$-in. corrugated round.	9	251	360	488

Comparing the values for the plain round rods with the results given in Table No. 8 it will be seen that for the $\frac{3}{4}$, $\frac{5}{8}$,

and 1-inch bars the average value by direct tension is about 45% greater than the results from the beam test. For the larger rods the difference is small. For the twisted square bars the results are also about the same. For the corrugated bar the maximum value corresponds to a slip of about .01 inch in the direct tension tests.

It was observed in the beam tests that the maximum resistance was reached for a very small end slip, about .001 inch, whereas in the tension tests the corresponding slip was about .01 inch. This appeared to be due to the progressive slipping action in the beam from centre or load points towards end, so that the average slip was a considerable amount when the end slip began. Some slip of bar necessarily occurs even under safe loads on account of the necessary stretch of bar to which the

TABLE No. 12.

BOND TESTS ON BEAMS.

(BACH.)

Concrete 1 : 4; age 6 months; beams loaded at quarter points.

No.	Kind of Beam.	Reinforcement.	Calculated Bond Stress at beginning of Slip, lbs/in².	Average for Group, lbs/in².
2	Rectangular Beams	Straight rods only	312	291
3			300	
4			271	
5			281	
161		Straight rods and stirrups	330*	330
162	T-Beams	Straight rods only	158	158
163		Straight rods and stirrups	182	195
164			208	
165		1 straight, 4 bent	408	493
166		1 straight, 4 bent, with stirrups	498	
167		1 straight, 4 bent, with stirrups	545	
168		1 straight, 4 bent	522	

* Average of three.

concrete adjusts itself by slight cracks at intervals, accompanied by more or less slipping. A very small end slip, therefore, indicates that the maximum bond resistance has been nearly reached.

122. Bond Stress in Beams with Bent Rods.—Table No. 12 gives results of test by Bach * on beams showing the effect of bending up some of the rods for shear reinforcement. The results with straight rods in the rectangular beams are about the same as previously given. In the T-beams with straight rods only they are rather low. The important results are those where four of the rods are bent leaving only one straight. Calculating the bond stress at the end of the beam by the usual formula, and taking into account only the one rod at the bottom of the beam, gives the values shown in the table, the average being 493 lbs/in². This is about $2\frac{1}{2}$ times the value for straight rods and stirrups. The same general result is shown in Table No. 18, Art. 126, where with some of the rods bent, a bond strength calculated for the straight rods, of over 600 lbs/in² was obtained. Inasmuch as the actual bond strength in these cases must have been practically the same as in other tests, these results show that where some of the bars are bent up so as to reinforce the web of the beam the bond stress on the remaining straight bars near the end of the beam is much less than the theoretical values obtained by the usual formula. Where such an arrangement of rods is used it is evident that the allowable bond stress on the straight rods may be very considerably increased.

123. Hooked and Anchored Ends.—It is a common practice where increased bond resistance is required, to bend the ends of rods into short hooks. Tests by Bach and Abrams show that right-angle bends increase the ultimate bond resistance about 50 to 60% while bends of 180 degrees give about double resistance. When hooked ends are used they should consist of bends through 180 degrees with a short length of straight rod beyond

* Mit. über Forsch. a. d. Gebiet des Ing., 1907, 45–47.

the bend. Such hooks are found to be very effective. Bars anchored by nuts and washers show about the same results as deformed bars; they require, however, a considerable slip before the anchorage develops its resistance. Inasmuch as a very small amount of end slip is sufficient to cause beam failures by diagonal tension, the end anchorage of straight bars gives but little better results than plain bars.

TESTS OF SHEARING STRENGTH OF BEAMS.

124. Before proceeding with a discussion of the methods of calculation and design of shear reinforcement it will be desirable to study the behavior of beams under test where the element of diagonal tension or bond stress was influential in the result. It will be convenient to consider separately tests on rectangular beams and those on T-beams.

125. Tests on Rectangular Beams.—(*a*) *Bars with Straight Reinforcement Only.*—Fig. 11 is a photograph of a typical shear

FIG. 11.—Shear Failure.

failure. Generally the first sign of failure is the appearance of the diagonal crack between the centre of the beam and the steel. As the load increases the crack extends diagonally upward to near the top of the beam. In most cases a longitudinal crack is formed just above the level of the bars, beginning at the diagonal crack and extending rapidly towards the end of the beam, the concrete stripping off from the body of the beam.

Table No. 13.

TESTS OF BEAMS REINFORCED WITH STRAIGHT RODS ONLY.

Shear Failures.

Authority and Reference.	Kind of Concrete.	Crushing Strength, lbs/in².	Net Cross-section, $b \times d$, Inches.	Span Length, Feet.	Kind of Bars.	Per Cent Reinforcement.	No. of Tests.	Shearing Stress at Failure $v = V/bjd$. Min. and Max. of Groups, lbs/in².	Shearing Stress at Failure $v = V/bjd$. Average of Group lbs/in².	Ratio of Shearing Strength to Crushing Strength.
Marburg, A.S.T.M., IV, 1904, p. 508	1 : 2 : 4 Age 30 da.	1700	7×8	5 and 8	Plain, cor. and twisted	.96-1.36	13	84–124	104	.061
Harding, *Jour. West. Soc.*, X, 1905, p. 705	1 : 2 : 5 Age about 4 mo.	1530	12×20, $12 \times 21\frac{7}{16}$, $12 \times 23\frac{1}{4}$, $12 \times 18\frac{3}{4}$	12	Plain round	.75	3	103–147	122	.080
				12	Corrugated	.72–.75	6	121–161	146	.096
				12	Twisted	.75	3	97–134	113	.074
Carson, Boston Transit Com., 1904.	1 : 2 : 3 approx. Age about 60 da.	4100	$6 \times 7\frac{1}{4}$	5	Corrugated	.82–1.64	6	203–265	246	.060
			$6 \times 7\frac{1}{4}$	5	Twisted	.9–2.3	4	175–262	220	.054
Talbot, 1907 Series, *Bull. Univ. of Ill.*, No. 29, 1909. Age about 60–70 days.	$1 : 1\frac{1}{2} : 3$		8×10	6	Plain round	.98-1.65	3	142–222	169	
	1 : 2 : 4		8×10	6 and 12	Pl. round and cor.	.98–1.53	21	89–190	140	
	1 : 3 : 6		8×10	6 and 12	Plain round	.98	4	72–98	82	
	1 : 4 : 8		8×10	6 and 12	Plain round	.98	6	62–90	73	
	1 : 5 : 10		8×10	6 and 12	Plain round	.98	4	49–84	65	
Talbot, 1908 Series, *Bull. Univ. of Ill.*, No. 29, 1909. Age 60–70 days.	1 : 1 : 2	4150	8×10	6 and 12	Plain round	.98–1.47	4	134–229	184	.044
	$1 : 1\frac{1}{2} : 3$	3315	8×10	6	Plain round	.98	2	142–181	161	.049
	1 : 2 : 4	2475	8×10	6 and 12	Pl. round and cor.	.98–2.21	24	128–214	164	.066
	1 : 3 : 6	1403	8×10	6	Plain round	.98	2	104–170	137	.098
	1 : 4 : 8	1296	8×10	6 and 12	Plain round	.98	6	65–113	90	.069
	1 : 5 : 10	940	8×10	6 and 12	Plain round	.98	7	37–82	63	.067

In Table No. 13 are given the results of a large number of tests on rectangular beams reinforced with straight rods only and which failed by diagonal tension failures. The shearing stress given in the table is the stress at failure calculated by the formula $v = V/bjd$. For 1 : 2 : 4 concrete this is seen to average about 140 to 160 lbs/in². The ratio of the shearing to the crushing strength of cubes ranges from about .05 to .07, averaging about .06. In the results of tests on T-beams in Tables 16 and 18, the shearing strength with straight rods only varied from 135 to 252 lbs/in².

In the tests by Talbot it was found that the strength in diagonal tension was less the longer the beam in proportion to its depth and also less the smaller the amount of reinforcement. This result is doubtless due to the increased stress and deformation in the horizontal steel for the longer beam and smaller amount of reinforcement. The effect of stress in the horizontal rods is also shown in a marked manner in a series of tests made at the University of Wisconsin on small mortar beams of 1·3 mixture. The beams were $3 \times 4\frac{1}{2}$ inches in cross-section and 4 feet span length. Loads were applied at two points a varying distance apart. Only straight reinforcement was used, amounting to 1.41%. The tensile strength of the material was high, being 490 lbs/in². The results were as follows:

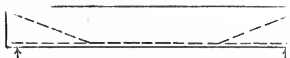
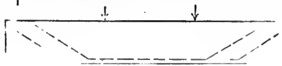
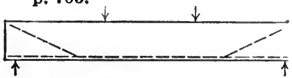
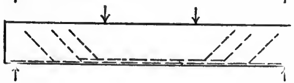
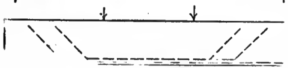
Distance Apart of Loads. Inches.	Maximum Shearing Stress. Lbs/in².
Centre Load.	205
12	232
24	255
32	365
36	595
40	990
44	1200

The increase in strength as the loads approached the supports must be due largely to the decrease in moment stress and

TABLE No. 14.

TESTS OF BEAMS REINFORCED WITH STRAIGHT AND BENT-UP RODS.

SHEAR FAILURES EXCEPT AS NOTED.

Authority and Reference.	Kind of Concrete.	Crushing Strength, lbs/in².	Net Cross-section, $b \times d$, Inches.	Span L'gth, Feet.	Kind of Bars.	Per Cent Reinforcement.	No. of Tests.	Shearing Strength at Failure $v = V/bjd$. Min. and Max. of Group, lbs/in².	Shearing Strength at Failure $v = V/bjd$. Average of Group, lbs/in².	Ratio of Shearing Strength to Crushing Strength.
Talbot, *Bul. Univ. of Ill.*, No. 29, 1909 	1 : 2 : 4 Age about 60 da.		8×10	6	2 bent, 2 straight...	.98–1.53	6	168–240	206	
	1 : 2 : 4 Age about 60 da.		8×10	6	Cumming's Welded Loops	1.32	4	201–266 Tension failure.	238	
Harding, *Jour. West. Soc.*, X, 1905, p. 705.	1 : 2 : 5(*a*)	1530	$12 \times 18\frac{3}{4}$	12	1 bent, 2 straight twisted	.75	3	175–179	176	.115
	(*a*) (*b*)		$12 \times 19\frac{3}{8}$	12	5 bent, 2 straight, twisted	.75	3	197–220	210	.137
	(*b*) (*c*)		$12 \times 21\frac{1}{16}$	12	3 bent, 2 straight cor.	.72	3	210–223	215	.140
	(*c*) (*a*)		$12 \times 23\frac{1}{4}$	12	1 bent, 2 straight cor.	.75	3	222–246	228	.149

consequent distortion, which is essentially what occurs when large areas of steel and low working stresses are used.

(*b*) *Beams with Bent-up Bars.*—Tests on beams with rods bent up at an angle give greatly varying results depending upon the manner of bending and the bond strength of the bars. Where all bars are bent up there is little or no gain in strength over the straight reinforcement, unless the ends of the bars are anchored, in which case there is some gain. Generally a part of the bars will be bent and a part left straight. Table No. 14 gives typical results of tests on beams reinforced in this manner.

From these tests it will be seen that the shearing-stress ranged from 175 to 246 lbs/in² for 1 : 2 : 5 concrete. As some of the failures were tension failures the actual shearing strength averaged somewhat higher than the values given.

Generally speaking, beams reinforced in this manner have shown deficient bond strength in the bent ends of the bars. Higher results can be obtained where the ends are securely anchored. Tests on the four beams with loops gave all tension failures indicating that the shearing strength was not reached.

TESTS OF BEAMS WITH STRAIGHT RODS AND VERTICAL STIRRUPS

UNIVERSITY OF ILLINOIS,* 1907.

Concrete 1: 2: 4; age about 60 days. Beams 8×10 ins.; span 6 ft.; loaded at third points. Shear failures. Longitudinal reinforcement; four ½-in. cor. bars=1.25%.

No. of Tests.	Stirrups.		Shearing Stress, $v = V/bjd$.	
	Kind.	Spacing, Ins.	Min. and Max. of Group lbs./in²	Average of Group lbs/in²
3	¼-in. cor. h. s. bars	6	242–272	255
4	¼-in. cor. m. s. bars	4	170–306	237
3	½-in. cor. m. s. bars	8	259–328	284
2	½-in. round bent inward at top.	8	224–246	235

* Bulletin No. 29, Univ of Ill, 1909.

(c) *Beams Reinforced with Stirrups.*—Table No. 15 gives results of tests on rectangular beams by Talbot, in which the horizontal reinforcement was of deformed bars and the stirrups either of deformed bars or plain bars anchored by bending. In a large number of the tests of this series in which plain round material was used for stirrups, the bond strength was insufficient.

126. Tests on T-Beams.—The reinforcing of T-beams requires special care in providing against shearing-stresses. Where a floor slab forms the upper part of a beam there will usually be ample strength in compression for any depth likely to be selected. The design of the stem of the T, or the beam below the slab, is therefore largely a question of providing sufficient concrete and reinforcement to take care of the shearing-stresses. In this case, therefore, it is important to provide a strong web for shearing-stresses, as the strength in this respect will commonly determine its size. In Tables Nos. 16 to 18 are given the results of tests on T-beams. The percentage of steel is calculated with reference to a rectangular beam having a cross-section equal to the circumscribing rectangle.

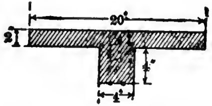
Referring to Table 16 it will be noted that of the five beams having straight rods and vertical stirrups, Nos. 4 to 8, No. 4 has stirrups spaced 8 inches apart, while the others have a spacing of 4 inches or less near the support. For the former a value of 233 lbs/in^2 was reached, while the three others averaged 393, all being nearly the same despite the variety of bars used. No. 9 had bent bars and no stirrups, giving a strength of 285, while No. 10 had bent rods and stirrups rather widely spaced, developing 380. Nos. 11–14 had inclined stirrups attached to the bars and all but the first gave high values of over 500 for the shear.

In those beams having .84% reinforcement, a load of about 19,000 lbs. would stress the steel to its yield point, so that apparently beams Nos. 12, 13, and 14 were already overstressed in tension.

Table No. 17 contains results of test on T-beams made

TABLE No. 16.

T-BEAM TESTS MADE BY F. VON EMPERGER.*



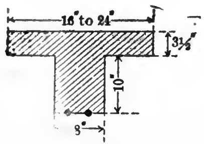
Concrete, 1:4; age, about 100 days.
Plain round steel used except where noted.
Beams loaded with two loads at $\frac{1}{4}$ points. Span = $78\frac{3}{4}''$.

Number.	Percentage Reinforcement.	Number and Approximate Size of Rods.	Arrangement of Rods.	Number and Spacing of Stirrups between Load and Support.	Total Breaking Load, Pounds.	Maximum Shearing Stress on Beam 4″×6″ Lbs/in².
1	.84	Two; .8-inch	Straight	None	5700	135
2	.64	Two Thacher bars, 7-inch	"	"	5700	135
3	.84	Eight; .4-inch	"	"	9470	225
4	.84	Two; .8-inch	"	3 double, spaced 8″	9800	233
5	.84	" "	"	5, spaced 4″	16000	380
6	.84	" "	"	6, spaced 2″ to 6″	16500	392
7	.64	Two Thacher bars, 7-inch	"	" " " " "	17200	410
8	.84	Eight; .4-inch	"	6 double, spaced $3\frac{1}{2}''$	15600	370
9	.84	" "	4 straight, 4 bent	None	12100	285
10	.84	" "	" "	4, spaced $5\frac{1}{2}''$	16000	380
11	.84	Two; .8-inch	Straight	4 inclined, spaced 6″	13800	327
12	.84	" "	"	4 double inclined, spaced 6″	21800	517
13	.84	" "	"	6 double, inclined, spaced 4″	23100	550
14	.75	Two; .32″×1.4″	"	" " " " "	23700	562

* Forscherarbeiten auf dem Gebiete des Eisenbetons, Heft V, 1906.

TABLE No. 17.

T-BEAM TESTS, UNIVERSITY OF WISCONSIN.*



Concrete 1:2:4; age, 28 days; compressive strength = 1940 lbs/in².
Beams D had 24-in. flanges, all others 16-in. Span length = 10 ft.; loaded at third points.
Stirrups uniformly spaced between loads and supports.
All beams failed in tension except G_1 and G_2, which failed by breaking of stirrups.

No. of Beam.	Kind of Reinforcement.	At First Diagonal Crack. Load, Lbs.	At First Diagonal Crack. Maximum Shearing Stress, lbs/in².	Tensile Strength of Concrete, lbs/in².	At Maximum Load. Load, Lbs.	At Maximum Load. Shearing Stress, lbs/in².	$\frac{M}{bd^2}$
A_1	Four ¾" cor. bars (2 bent).....	24000	128	256	66600	361	452
A_2	Fourteen ¼" cor. stirrups......	32000	171	167	66200	357	450
B_1	Four ¾" cor. bars (2 bent).....	42000	226	260	65600	357	444
B_2	Sixteen ¼" round stirrups......	26000	150	157	62400	354	425
C_1	Five ¾" round rods (3 bent)....	34000	181	197	60000	322	407
C_2	Sixteen ¼" round stirrups......	34000	185	216	57400	316	390
D_1	Six ¾" cor. bars (3 bent)	46000	247	182	96200	526	437
D_2	Fourteen ⅜" cor. stirrups......	58000	306	...	101400	544	460
E_1	Six ¾" cor. bars (3 bent).......	46000	244	148	92800	505	420
E_2	Twenty-four ⅜" cor. stirrups...	40000	215	184	88000	485	400
F_1	Four ¾" cor. bars (2 bent).....	30000	158	142	67600	368	459
F_2	No. 11 wire mesh, 1 inch.......	28000	150	174	65400	352	445
G_1	Four ¾" cor. bars (straight)....	24000	126	184	48000	259	323
G_2	Sixteen ¼" round stirrups......	28000	149	164	48200	260	324

* Bulletins No. 2, Vol. 4, 1908.

at the University of Wisconsin. In these tests the yield point of the corrugated bars was about 48,000 lbs/in² and that of the $\frac{3}{4}$-in. rods was 41,000 lbs/in². These limits correspond closely to the stresses in the steel at failure, excepting in the case of beams G_1 and G_2 which failed by shear. The table contains results of value with respect to shearing-stresses and the use of stirrups and bent rods for shear reinforcement. In the progress of the tests the occurrence of the first diagonal crack was carefully noted and the maximum shearing-stress at this load is calculated and given in the table. It will be noted that there is a fairly close agreement between this value and the tensile strength of the concrete as given in the next column. The average value for the maximum shearing-stress is 179 lbs/in² whereas the average tensile strength is 187 lbs/in². This would indicate that in spite of stirrups the concrete is likely to open up at a diagonal tensile stress about equal to its usual tensile strength. The table also gives the maximum shearing stress at ultimate load. The web reinforcement was effective in preventing shear failures, excepting in the case of beams G_1 and G_2 where no bent rods were used. In all other cases the web reinforcement was ample and hence no conclusions can be drawn as to relative value of the different kinds of reinforcement. The wire mesh gave good reinforcement and was convenient to use. In beams G_1 and G_2 failure by diagonal tension occurred, the stirrups breaking at the maximum load. These tests indicate that by the use of bent rods and stirrups a shearing strength of 450 lbs/in² can readily be developed with concrete of the quality here used.

Table No. 18 contains results of tests by Bach in which a variety of forms of reinforcement was used. Comparing Nos. 1 and 3, it appears that hooking or anchoring the ends of the horizontal rods is not of much value where no web reinforcement is used; the bond strength of the plain rods is apparently sufficient to develop the shearing strength of the concrete. However, Nos. 2 and 4 show that when vertical stirrups are used, hooks are advantageous. They are less so

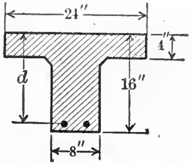


TABLE NO. 18.

T-BEAM TESTS BY BACH.*

Compressive strength of concrete in cubes = 3320 lbs/in².
Yield point of steel = 46,000 lbs/in².
Reinforcement at center = approximately 1.2%.
Stirrups made of round rods .28 inches in diam., excepting No. 6, in which flat strips were used .08″ × .8″ in section.
Span length, 13′–2″; loaded at 8 points uniformly spaced.
Each result is the average of three tests.

Type of Reinforcement.	No.	Calculated Stresses at Rupture.				Kind of Failure.
		Tension in Steel, lbs/in².	Compression in Concrete, lbs/in².	Shearing Stress, lbs/in².	Bond Stress in Horizontal Rods, lbs/in².	
	1	22,400	1310	252	203	Shear.
	2	32,000	1870	360	290	Shear and Bond.
	3	24,400	1420	273	230	Shear.
	4	43,300	2550	495	394	Shear.
	5	37,000	2150	405	465	Shear.
	6	50,400	2930	550	640	Tension.
	7	41,800	2500	470	600	Tension.
	8	46,300	2770	520	660	Tension.
	9	50,500	2960	568	610	Tension.
	10	49,400	2900	552	800	Tension and shear.

* Deutscher Ausschuss für Eisenbeton. Heft 20, 1912.

where rods are bent up (compare Nos. 7 and 8). Thorough reinforcement by stirrups (No. 4) gives as good results as bent rods alone (No. 7), unless spaced very closely (No. 10), but not quite as good as a combination of bent rods and stirrups (Nos. 6 and 9). Bent rods, even when few in number, add much to the strength (No. 5). Finally, the shearing values are about the same as shown in previous tables and it appears with the quality of concrete here used a shearing strength of at least 550 lbs/in² can be secured. Compared to the crushing strength of the concrete this is not quite as high a value as shown in Table No. 10. Inasmuch as Nos. 6 to 10 failed in tension, the maximum shearing strength was not reached in these cases.

127. Development of Cracks under Tests.—An important question pertaining to the use of various kinds of shear reinforcement is the manner of development of cracks when tested to destruction. According to the analysis of Art. 116 vertical stirrups are not in a position to be stressed in tension until the concrete begins to deform more in tension than in compression, and they cannot take much stress until cracks begin to form. Inclined web members, bent bars or stirrups, can, on the other hand, take some stress before the concrete begins to crack, but the amount of such stress is not large. On this subject the tests of Bach are very instructive as the development of cracks in each beam are shown for several stages of the tests. In Fig. 12 are shown the behavior of four of the beams tested. The kind of reinforcement is shown and the total load in kilograms.

Comparing A and B the results are seen to be about the same. More important are the results shown in C and D. In C there are vertical stirrups only, while in D the beam is very thoroughly reinforced by both stirrups and bent rods. Notwithstanding this difference the development of cracks is nearly the same in both cases, but D is stronger than C. At a load of 12,000 kg. all beams show considerable cracking and about as much in those fully reinforced as in A and B. This load cor-

responds to a stress of about 12,000 lbs/in² in the steel. It may be said, therefore, that no kind of web reinforcement will

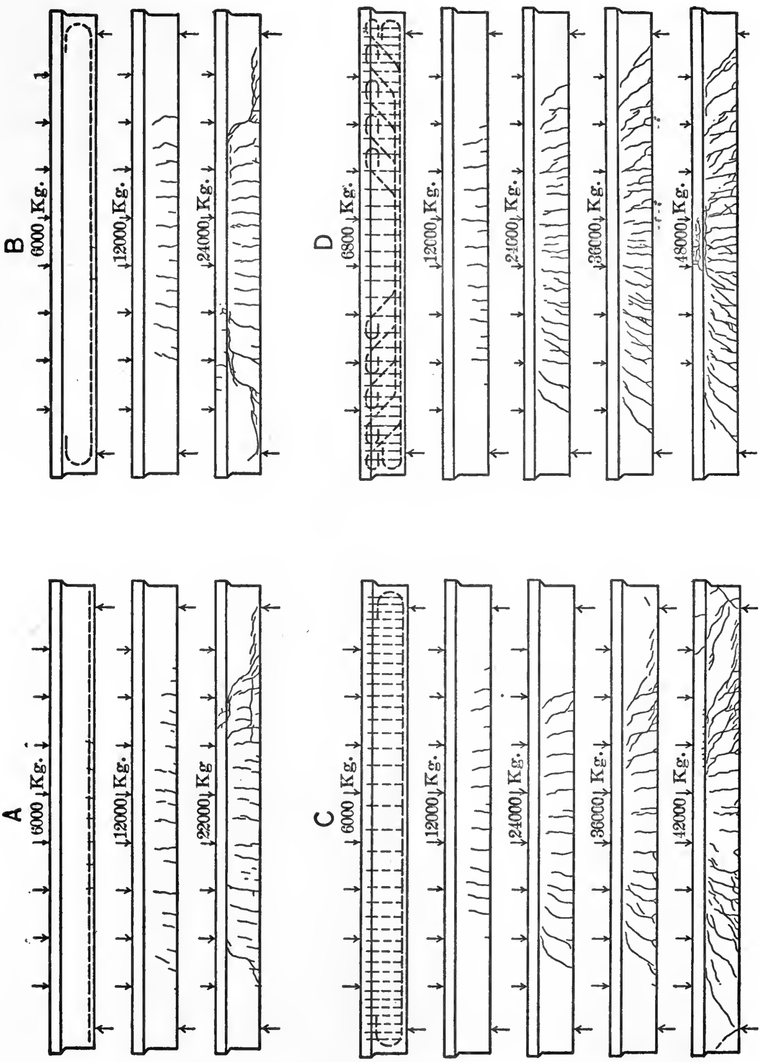


Fig. 12.—Development of Cracks in T-beams.

prevent incipient cracking. What web reinforcement accomplishes is to prevent diagonal cracks from opening up to a danger-

ous extent and greatly to increase the ultimate strength of the beam. In this respect the vertical stirrup appears from these tests to be about as effective as bent rods.

128. Conclusions as to Shearing Strength.—From the available data it would appear that with ordinary concrete and no web reinforcement the ultimate shearing strength is about 125 to 150 lbs/in^2 and that this strength can readily be increased by the use of web reinforcement to 350 to 450 lbs/in^2. The latter figure may be taken as about the maximum value with ordinary, closely-spaced web reinforcement.

These and other tests show the necessity of carefully providing for the bond stress in stirrups and in the bent ends of rods. The shearing strength is evidently affected considerably by the stresses in the horizontal reinforcement, a high stress resulting in a lower shearing strength. Vertical stirrups do not come much into action until a slight diagonal crack begins to form, after which they begin to act. They are, however, about as effective as bent rods. Failure by diagonal tension in beams without web reinforcement is especially objectionable, as such failure is likely to occur suddenly and without warning. Web reinforcement not only strengthens the beam, but causes a slow failure.

It is to be understood that the shearing-stress is here used merely as a convenient measure of the diagonal tensile stress, which is really the stress involved. This being the case it would be incorrect to take any account of the shearing strength of the steel in designing the reinforcement, as is sometimes done.

CHAPTER V.

DESIGN OF BEAMS.

129. Working Stresses and Factors of Safety.—In the design of steel structures it has come to be the practice to make use of definite working stresses rather than factors of safety. These working stresses are based, for the most part, on the permanent elastic-limit strength of the material, although the margin of safety between the elastic-limit and the ultimate strength (indicated by strength and ductility) receives consideration. The working stresses are made sufficiently below the elastic limit to provide for:

(*a*) Variations and imperfections in material and workmanship.

(*b*) Uncalculated stresses, such as secondary stresses, stresses due to unequal settlement, and, usually, those due to temperature changes.

(*c*) Dynamic effect of live load if not provided for by an allowance for impact.

(*d*) Possible increase in live load over that assumed, or rare applications of excessive loads.

(*e*) Deterioration of the structure.

The more accurately the various elements are determined in any case the closer may the working stress approach the elastic limit. Where the dynamic effect of the live load does not enter, or is otherwise fully provided for, and where items (*d*) and (*e*) are of small moment, working stresses for steel structures will vary from about one-half to two-thirds the elastic-limit strength of the material. Were it absolutely certain that

the elastic limit of the material would never be exceeded in any emergency, then the margin of strength between the elastic limit and the ultimate strength would be of no importance. This is, however, not the case, and under actual conditions of construction and use there is a very important element of safety in the fact that the ultimate strength is, in most materials, much higher than the elastic limit. The ductility of the material, or its possible deformation beyond the elastic limit, is also of great importance. While, therefore, the working stresses are selected chiefly with reference to the elastic limit, the ultimate strength also receives consideration.

In reinforced-concrete design the problem is complicated by the use of two unlike materials whose elastic limits and ultimate strengths are not similarly related. Furthermore, as the materials are stressed beyond their elastic limits the stresses do not necessarily increase in proportion to the load, so that if working stresses of one-fourth the ultimate are selected, for example, the corresponding load may be considerably greater or less than one-fourth the ultimate load. This condition makes it especially desirable to consider ultimate strengths as well as elastic limits.

130. Relative Effect of Dead and Live Loads.—The tendency of practice in the treatment of live-load stresses is to reduce them to equivalent dead-load stresses by the application of some sort of impact formula or by other means of estimation. The resulting stresses are then considered on the same basis as the usual dead-load stresses and a single set of working stresses applied. This method is simple, logical, and tends to facilitate a proper adjustment of the design to the conditions. Separate working stresses will give equally satisfactory results when properly selected, but the system is not as flexible or convenient as the method of the single working stress with impact coefficients.

The question of impact coefficients, or the relation between live- and dead-load working stresses, requires little special attention in connection with reinforced concrete structures.

It is essentially the same as it is in the case of steel structures, excepting as the amount of impact may be modified by the structure itself. In steel railroad structures of short span, for example, the impact, or dynamic effect of live load, is usually assumed to be about 100% of the live load stresses. Experiments show that this is probably not too high and that the actual stresses from live load may be 100% greater than the static stresses, due largely to the effect of unbalanced locomotive wheels. Where a large amount of ballast intervenes between the load and the structure the impact is doubtless much less. In the case of concrete structures the great mass of the concrete undoubtedly tends to reduce the effect of impact and vibration, or to localize such effect more than in a steel structure. The conditions involved in concrete designing, therefore, are likely to be favorable as regards impact and may permit the use of lower coefficients than are used for steel structures. The proper coefficient to use, or the relation between live- and dead-load working stresses, varies much under different conditions and must be left to the judgment of the designer, or to formulas or rules prepared especially for the purpose. Further discussion of this question will not be undertaken here.

In buildings it is the practice in steel construction to use a single working stress, no account being taken directly of any special effect of the live load. Allowance is made in the design of large girders and columns which receive their load from large areas for the fact that such large areas, especially if on two or more floors, are seldom or never loaded to the extent assumed for smaller areas. This allowance varies with different conditions, but relates solely to the selection of the amount of live load rather than to its effect. In a building, when heavily loaded with its live load, the portion of the load which is in motion and capable of producing a dynamic effect is generally but a very small percentage of the total live load. In most cases, therefore, in building construction it is not necessary to treat the live-load stresses differently from the dead-load stresses, and the design is based on a single set of working stresses.

Special cases will arise, however, where the dynamic effect of the live load requires consideration, as, for example, in the case of floors supporting moving machinery.

Whatever the effect of live load may be it can more readily be taken account of by adding to the resulting live-load stresses a percentage which, in the judgment of the engineer, will reduce them to their dead-load equivalent, and then apply a single set of working stresses, or factor of safety, to the sum of the stresses. The discussion of working stresses in the following articles will relate to the proper basal working stress for dead load, or for live load suitably increased for impact.

131. Working Stresses in Tension and Compression.—The strength of a beam is limited usually by:

(*a*) The compressive strength of the concrete,
(*b*) The elastic-limit strength of the steel, or
(*c*) The strength of the beam in diagonal tension.

In this article the first two elements only will be considered.

From tests relative to elastic limit, such as those of Bach and Van Ornum (Art. 27), it would appear that the permanent elastic limit of concrete is from 50 to 60% of its ultimate strength as determined in the usual manner. If a factor of safety of two be applied to the elastic-limit, we will have a dead-load basal working stress of 25 to 30% of the ultimate strength as determined by tests on cubes. Taking this ratio at 30%, the data of Chapter II show that the working stresses for concrete of the usual proportions (1 : 2 : 4 to 1 : $2\frac{1}{2}$: 5) should range from about 550 to 650 lbs/in⁴. A value of 600 lbs/in² is commonly used and should imply a strength of about 2000 lbs/in² in cube form in 60 days. As shown in Art. 67 when a beam is progressively loaded to its breaking point, the stress in the concrete does not increase in proportion to the load on the beam, so that a working stress of 30% of the compressive strength will give a factor of safety against failure of about $4\frac{1}{2}$. It is to be noted also that the strength of the concrete increases with age. On the whole, therefore, a stress of 30%

of the strength at 60 days may be considered as conservative practice.

With respect to the steel it is to be observed that its elastic limit, or more correctly speaking, its yield point, determines not only the elastic limit strength of the beam, but, also approximately, its ultimate strength, and the working stress should be selected with this in view. If, for example, the working stress is taken at one-half the elastic limit strength of the steel, the factor of safety will be two as to elastic strength and about $2\frac{1}{2}$ as to ultimate strength. With a working stress in the concrete of one-half its elastic limit the beam will than have a factor of safety as regards elastic limit of about two (determined by the concrete), and as regards ultimate strength its factor of safety will be at least five relative to the concrete and about $2\frac{1}{2}$ relative to the steel. Its elastic limit is thus determined by the concrete and its ultimate strength by the steel, which may be considered as satisfactory conditions. The greater uniformity and reliability of the steel, as compared to the concrete, should be noted in this connection.

The working stresses in the steel should also be considered with reference to its distortion. High working stresses involve large distortions, and hence a greater degree of incipient rupture in the concrete. The deformations of the concrete are also of importance with reference to their effect on diagonal tensile stresses, as explained in Art. 113. Low unit stresses in the steel are greatly to be preferred on this account. Considering these facts, it would seem that a stress of 16,000 lbs/in^2 should be considered about the maximum desirable value, irrespective of the quality of the steel used.

As a result of this analysis, therefore, we may conclude that the basal working stress in the steel should not exceed about one-half its elastic limit nor exceed about 16,000 lbs/in^2.

132. Working Stresses in Shear.—From the results given in Chapter IV, the ultimate diagonal tensile strength of a beam having no web reinforcement, as measured by the maximum shearing stresses developed, may be taken at about 120 lbs/in^2.

Inasmuch as a failure due to high shearing stresses is apt to be sudden, the factor of safety should be at least three. This gives a working stress of about 40 lbs/in². For beams in which the web is well reinforced the working stresses may be made as much as three times as great, or about 120 lbs/in². The results of tests noted in Art. 126 show that in spite of ample web reinforcement visible cracks will form in the webs of beams at maximum shearing stresses about equal to the ultimate tensile strength of the concrete, or about 180 lbs/in² in the beams tested. The working stresses should therefore keep well within this limit.

The stresses here considered relate to shearing-stresses involving large diagonal tensile stresses. Where such tensile stresses are not developed to any extent, as in "punching" shear, a higher value may be employed; but as it is almost impossible in practice to avoid altogether such tensile stresses it is not advisable to greatly increase the working stresses above the maximum value of 120 lbs/in² above suggested.

133. Working Bond Stresses.—The factor of safety with reference to the slipping of the rods should be at least three, since the strength of a beam should not be limited by the strength of bond. From the data of Chapter IV, we may take the bond strength of plain steel at from 250 to 350 lbs/in². A working stress of from 75 to 90 lbs/in² is therefore suitable. Increase in age will increase the factor of safety in this respect very considerably. With a working bond stress of 75, for example, and a tensile unit stress of 16,000 lbs/in² a round bar will need to be imbedded a length of $16{,}000/(4\times75)=53$ diameters to develop its full strength, assuming uniform bond resistance. In the case of large bars of 1 to $1\frac{1}{2}$ inches in diameter this length is very considerable, and for short beams may be difficult to secure. The deformed bar or the anchored bar is of special value under these conditions.

Where some of the bars are bent up near the end of the beam, a considerably higher unit stress may be used if only the straight rods be included in the calculations. The tests

mentioned in Art. 122 indicate that for conditions comparable to those existing in the tests in question, the allowable working stress may safely be increased 50 per cent.

For deformed bars the working stress may be somewhat greater than for plain bars, but as the initial slip occurs at about the same load, the increase in working stress should not be large.

134. Working Stresses Recommended by the Joint Commitee. The following working stresses for beams are recommended by the Joint Committee on Reinforced Concrete:

Compression in Concrete.—Extreme fibre stress, f_c, 32.5% of the compressive strength of the concrete.

Tension in Steel.—16,000 lbs/in^2.

Shear as a Measure of Diagonal Tension.—(*a*) For beams with horizontal bars only, 2% of compressive strength;

(*b*) For beams with vertical stirrups or bent rods properly spaced, $4\frac{1}{2}$% of compressive strength;

(*c*) For beams with vertical stirrups and bent rods, 5% of compressive strength;

(*d*) For beams with attached stirrups, with or without bent rods, 6% of compressive strength.

Bond.—For ordinary plain bars, 4% of compressive strength; for drawn wire, 2% of compressive strength; and for the best type of deformed bars, 5% of compressive strength.

135. General Arrangement of Reinforcement.—Fig. 7 of Art. 115 illustrates various common arrangements of reinforcement for beams of one span. These are applicable to both rectangular and T-beams. Figs. (*a*) and (*b*) are suitable for small loads and beams of small depth where the shearing-stresses are so low that web reinforcement is unnecessary. If it is desired to use relatively few and large rods, the bond strength of Fig. (*a*) may be insufficient, requiring the use of hooks as in Fig. (*b*). For shearing-stresses somewhat above the strength of the concrete, Fig. (*c*) may be used. Here a portion of the rods are bent up in a single long bend The shearing strength of this type is increased at least 25% over the beam with straight

rods only. The addition of stirrups near the end, Fig. (*e*), as in the Hemibique system, further strengthens the beam and is an arrangement comparable to other methods of web reinforcement used to develop the maximum shearing strength. Fig. (*d*) shows straight rods and vertical stirrups; Fig. (*h*), inclined stirrups attached to the horizontal rods; Fig. (*f*), the use of bent bars, closely spaced; and Fig. (*g*), a combination of bent bars and stirrups. For the best results the combination shown in Fig. (*g*) is usually employed. Tests indicate that the strength of type (*d*) is somewhat less than (*g*) and it is not usually convenient to bend up a sufficient number of rods to make type (*f*) equal to (*g*). Type (*h*) requires special forms of rods and stirrups, but is thoroughly efficient.

In all cases but (*a*) the horizontal rods are shown hooked at the ends. This detail is very desirable. Even when the calculated bond stress is not high, results of tests show considerable gain in ultimate strength from such a detail where the strength of the beam is determined by its strength in diagonal tension.

136. Size, Length, and Spacing of Horizontal Bars.—The total required section is determined by the moment. The desirable size of horizontal rod to use will generally range from $\frac{3}{4}$ inch, as an ordinary minimum for small beams, to $1\frac{1}{2}$ inches as a maximum diameter for very large beams. Sizes larger than $1\frac{1}{2}$ inches are inconvenient to handle and a diameter less than $\frac{3}{4}$ inch is also inconvenient to use except in very small beams where the required area demands less. Between these limits the size depends upon the number and arrangement of rods desired. If all rods are to be straight, then a minimum number which will give the necessary bond strength at the ends may be used. For equal total area the larger the number of rods the greater the total circumference and hence the greater the bond resistance. To furnish the desired bond resistance requires, therefore, that the diameter of the rods shall not be too great. If a portion of the rods are to be bent up to reinforce against shear, then the size and number of the straight

rods must still be determined on the basis of bond strength. The number of rods to be bent up will depend upon the strength required in shear, and whether all of this is to be supplied by bent rods or only a part. To be fully effective rods should be bent up at intervals apart about equal to the depth of the beam, but for large and heavy beams this may require too many for convenience, and stirrups are generally used to aid. Rods are generally bent up at two or three points only.

137. *Lengths of Bent-up Bars.*—The minimum lengths of the several horizontal bars are determined by the bending moments to be carried. In determining these lengths the same methods, either analytical or graphical, may be employed as in the design of plate girder flanges. If the moment is due to a uniform load, the parabolic formula may be used. The lengths of the several rods are given by the equation

$$x_n=\frac{l}{\sqrt{A}}\sqrt{a_1+a_2+\ldots+a_n}, \qquad (1)$$

in which x_n = length of the nth rod in order of length counting the shortest one as number one;

l = length of span;

A = total steel area at center;

$a_1+a_2+\ldots+a_n$ = sum of areas of all rods up to the one in question (the nth rod).

If the amount of steel actually used is considerably greater than required, then the formula may be modified by making A = required area and deducting the excess from the area of the first rods. Thus,

$$x_n=\frac{l}{\sqrt{A_r}}\sqrt{a_1-(A_n-A_r)+a_2+\ldots+a_n}, \qquad (2)$$

in which A_r = area required;

A_n = area used.

For unsymmetrical or concentrated loading the actual moment curve must be determined and the length of bars determined therefrom, as in plate girder design.

The required theoretical lengths having been found the rods may be bent up at these points or at any desired place between these points and the supports. Or, if not bent up, they may be discontinued a few inches beyond the theoretical points and the ends bent into hooks for better bond.

138. *Spacing of Bars.*—In rectangular or T-beams the spacing of bars is important; in T-beams this consideration will largely control the width of the beam. The requirement in general as to spacing is that the bars must be spaced sufficiently far apart to readily admit the concrete between and beneath them and to give sufficient section along the plane of the rods to prevent failure by tension or shear. In the case of round rods, the weakest horizontal section relative to shear will be somewhat above a diametral plane (assuming uniform bond stress around the periphery of the rod). If u = bond stress and v = shearing stress, the required spacing c. to c. of rods when placed in a single layer is given by the equation $a = d\cos\theta + \frac{u}{v}d\left(\frac{\pi}{2}+\theta\right)$, in which d = diameter of rods and θ is given by the equation $\theta = \sin^{-1}\frac{u}{v}$. For $u=v$, $a=3.1d$; for $u=\frac{3}{4}v$, $a=2.35d$, and for $u=\frac{1}{2}v$, $a=1.9d$.

In the case of square bars with sides vertical, the *clear* spacing must equal $3d\frac{u}{v}$, and for their sides placed diagonally, $a=4d\frac{u}{v}$, or $a=d\left(1.4+\frac{2u}{v}\right)$, whichever gives the greater value. For more than one layer these formulas give the spacing in the bottom layer. The spacing in the upper layers will need to be greater as the shearing stress results from the action of all the rods below.

But in addition to the shearing stresses there is likely to be developed more or less tension in the concrete surrounding the rods, so that there should be left ample areas of concrete between

them, especially towards the end where the bond stresses are large. The space should also be sufficient to permit satisfactory manipulation of the concrete. A minimum clear spacing of at least $1\frac{1}{2}$ diameters should be provided, with an equal distance between the outside rod and the surface of the beam. The Joint Committee requires 2 diameters in both cases. Where some of the rods are bent up the spacing can readily be made more liberal towards the end of the beam. Between two horizontal layers of rods the spacing may be less but should be sufficient to insure good bond.

Liberal spacing, or large net section of concrete, favors large rods and few in number; good bond strength without waste of material favors small rods. If bent rods are to be used for web reinforcement, then numerous small rods are also advantageous. If the bond strength is not in question, or can easily be taken care of, then large rods are desirable, but more stirrups or other secondary reinforcement may be needed than where small rods are used.

139. Design of Shear Reinforcement.—The general action of shear reinforcement in resisting diagonal tensile stresses has been explained in Art. 116 and illustrated by the results of tests given in Arts. 125, 126. It has been shown by these tests that beams of ordinary concrete of about 1 : 2 : 4 proportions, without shear reinforcement, show an ultimate strength in diagonal tension as measured by the shearing stress of about 125 to 150 lbs/in². A safe value for the shearing-stress for such beams is about 40 lbs/in². Where a higher stress must be carried, some form of shear reinforcement is required, and a method of estimating the stresses in such reinforcement must be applied. As already described, the usual reinforcement consists of: (1) vertical stirrups, (2) inclined stirrups attached to horizontal bars, and (3) bent-up bars. Owing to the complex nature of the forces acting, it is not possible to calculate the stresses in such reinforcement with any high degree of accuracy, but a rough estimate of the requirements can be determined on rational grounds.

140. *Relative Proportion of Shear Carried by Concrete and Steel.*—In estimating the stresses in the reinforcement an important question arises as to the mutual action of concrete and steel, and whether the concrete can still be counted upon to carry a portion of the stress or whether the steel must be proportioned to carry the entire load. Some estimate of the actual stresses coming upon stirrups can be obtained by an analysis of the results of the tests described in Arts. 125, 126. Calculating the stresses in the stirrups of Talbot's tests (Table No. 15) by the formula of Art. 141, we find that these stresses at failure amount, in some cases, to as much as 100,000 lbs/in². These calculated stresses are beyond the ultimate strength of the material and hence are greater than the actual stresses.

From tests by Professor Withey * it was found that in the the case of two rectangular beams and two T-beams reinforced by horizontal bars and vertical stirrups, failure was caused by the overstressing of the stirrups, in two cases the stirrups breaking. The results were as follows:

No. of Beam.	Cross-section of Stirrup. Sq. in.	Spacing of Stirrups. Ins.	Net Depth of Beam. Ins.	Width of Beam. Ins.	Max. Shearing Stress. Lbs/in².	Calculated Stress in Stirrup. Lbs/in².
G_1	.049	$5\frac{1}{2}$	$13\frac{1}{2}$	8	259	117,000
G_2	.049	$5\frac{1}{2}$	$13\frac{1}{2}$	8	260	118,000
M_1	.049	6	16	8	335	164,000
M_2	.049	6	16	8	290	142,000

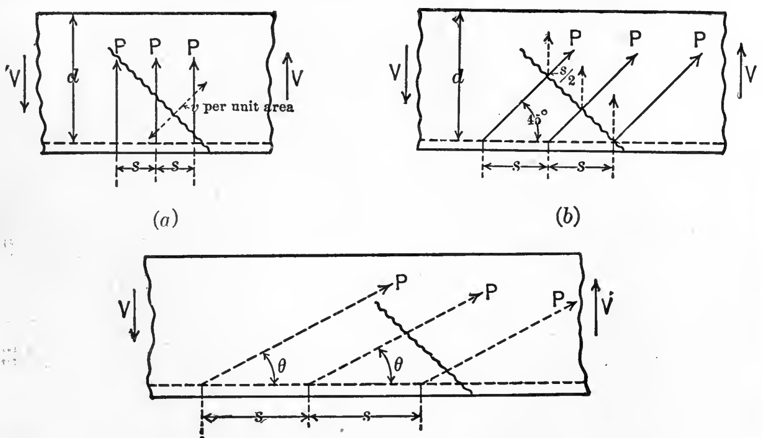
The stirrups were single loops of $\frac{1}{4}$-inch round steel having a yield point of 47,000 lbs/in² and an ultimate strength of 62,000 lbs/in².

The calculated stresses were determined from the formula of Art. 141, assuming that all the shear is carried by the stirrups. As these stresses are much beyond the ultimate strength of the stirrups, it is evident that a large amount of shear (about 50%) was carried by the concrete and by the bending resistance of the horizontal rods. Tests on beams without stirrups show

* Bull. No. 2, Vol. IV, 1908, Univ. of Wis.

the average shearing strength of concrete to be about 100 lbs/in², indicating that approximately correct results would be reached if the concrete be assumed to carry its full value and the stirrups the remainder. Similar results have been reached by other experimenters. If this is true at ultimate loads, it would be even more certain at working loads where the concrete is only slightly cracked and the distribution of stress more normal. From these considerations we may conclude that in calculating stresses in web reinforcement the concrete may be assume to carry a considerable proportion of the shear, and the steel then proportioned to carry the remainder. The Joint Committee recommends that where shear reinforcement is used, one-third the total vertical shear be assumed as carried by the concrete, and the reinforcement be designed for two-thirds.

141. *Formulas for Stress in Shear Reinforcement.*—Fig. 1 represents two forms of reinforcement, vertical and inclined



(a) (b) (c)

FIG. 1.

at 45 degrees, the latter being either attached stirrups or bent bars. Let s represent the horizontal spacing in both cases and assume a line of failure at 45 degrees. Let v = intensity of shear to be carried by the reinforcement (total shear less

that assumed as carried by concrete). This will also be taken as the intensity of the diagonal tensile stress at 45 degrees.

In the case of vertical stirrups they will be called upon to carry the vertical component only of this diagonal tension the horizontal component being carried by the horizontal bars. This vertical component per unit of horizontal area will be v. Assuming the same stress in each stirrup, represented by P, we have

$$P = vbs. \qquad (1)$$

For inclined stirrups or bent rods at 45 degrees, Fig. (b), the spacing at right angles to the line of rupture is $s \cos 45°$ and the horizontal spacing along the line of rupture will be $s \cos^2 45° = \frac{1}{2}s$. The vertical component of stress will therefore be, as in eq. (1), $.5vbs$, and the stress itself will be

$$P = .7vbs. \qquad (2)$$

For stirrups inclined at other angles, Fig. (c), the same reasoning leads to the general formula

$$P = \frac{vbs}{\cos\theta + \sin\theta}, \qquad (3)$$

where θ = inclination of stirrup with the horizontal and s = spacing measured horizontally. In eqs. (1), (2), and (3), the value of b is the width of beam in the case of a rectangular beam, and width of web in case of a T-beam.

142. *Spacing and Other Details.*—In arranging the details of shear reinforcement of large beams, it is generally convenient to select a certain size of stirrup and then calculate the necessary spacing at various points along the beam. If such spacing is too close or too far apart, a larger or smaller stirrup can be tried until a satisfactory arrangement is obtained. The maximum allowable spacing of bent bars must also frequently be determined. Both of these problems are simple, and require merely the use of eqs. (1), (2), or (3), solved for s.

To be reasonably effective the web reinforcement should be

so spaced that at least one rod will intersect any 45-degree line of rupture below the center of the beam. As shown by the sketch, Fig. 2, this requires a spacing of vertical reinforcement not greater than $d/2$, and for diagonal rods, a horizontal spacing not greater than d. Considerable gain in strength is obtained by rods spaced somewhat further apart, but tests show little value from vertical rods spaced a distance apart equal

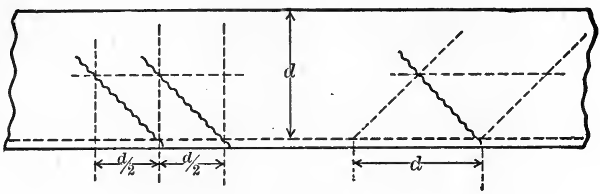


FIG. 2.

to d. The Joint Committee recommends a spacing of $\frac{3}{4}d$ for inclined stirrups and bent rods. The working stress to be used in these calculations should be no higher than permitted elsewhere, and preferably lower, as it is desired to prevent large distortions so far as practicable. In bending up horizontal rods those remaining straight should be ample in number to take the moment stresses, and, preferably, at reduced intensities towards the ends of the beam. It will often be impracticable to provide as much reinforcement as desired by means of bent-up rods, and some vertical stirrups will be needed, especially near the end where the stresses are high. A combination of bent rods and vertical stirrups is common practice and readily lends itself to adequate and convenient treatment. For large beams, under heavy shearing-stresses, both should be used. Inclined stirrups are quite as effective as bent rods or vertical stirrups, but to prevent slipping on the horizontal rods they must be securely fastened thereto. When so fastened, they are very efficient, and increase the bond strength of the horizontal rods.

The exact inclination of inclined reinforcement is not important. An inclination of less than 45 degrees will accord

more nearly with the direction of maximum stress near the bottom of the beam than a 45-degree slope, especially towards the centre of the beam. Considering, however, the fact that the horizontal rods are in position to carry all horizontal stress, a flat slope of the web reinforcement is unnecessary. At the ends a 45-degree slope is about right and rods are commonly bent at this angle.

The bond strength of the shear reinforcement must be carefully guarded, especially in the case of large bent-up rods. This strength should be provided in the upper portion of the beam above the neutral axis. Plain and bent-up rods often lack sufficient bond strength to render them fully effective. In non-continuous beams the ends of the bars should be bent into hooks.

It should be carefully noted that the stresses in the steel reinforcement here dealt with are tensile stresses, not shearing-stresses. When a beam fails in diagonal tension the horizontal rods are stressed somewhat in shear (See Fig. 5, Art. 113) but without stirrups this shearing resistance is of little value, as the bars readily bend and strip the concrete off from the bottom of the beam. The shearing strength of the horizontal rods should not be taken into account in the calculations.

143. Provision for Bond Strength.—In calculating bond stress the method of Art. 110 should be used. This theoretical analysis assumes that there is perfect adhesion and that the stress in the steel is taken over by the concrete at all points without slip. The bond stress is a simple linear function of the shear and varies therewith. As a matter of fact the actual bond stress is shown by tests to vary considerably from the theoretical results. It is much modified by the necessary slight slip of the rods at various points due to stretch under tension, which is not uniform, and is modified by the presence of web reinforcement. Stirrups tend to concentrate the bond stress near them, and bent-up rods modify greatly the bond stress in the horizontal rods. The tests by Bach (Art. 122) show that with bent-up rods the calculated bond stresses at first

slip in the remaining horizontal rods is about double the value with straight rods only. This difference may well be taken into account in design.

144. *Diameter and Spacing of Bars with Reference to Bond Stress.*—Bond resistance is an important factor in determining the size and number of straight rods necessary near the end of a beam. For equal sectional areas, the smaller the rods the greater the bond resistance. It will be helpful to establish a relation between the bond stress and the diameter and spacing of rods.

From eq. (2), Art. 110, the unit bond stress is $u=\dfrac{V}{jd\cdot\Sigma o}$. Also the maximum shearing stress is $v=\dfrac{V}{bjd}$. Combining, we have

$$u=\frac{vb}{\Sigma o}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a convenient relation between unit bond stress and unit shear. For given values of v and u we have

$$\Sigma o=\frac{vb}{u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, the necessary total perimeter of rods must be equal to $\dfrac{vb}{u}$, where v is the actual maximum shearing-stress at the section, u = allowable bond stress, and b = width of beam. In terms of the diameters of round rods we may write, from the above,

$$\Sigma D=\frac{v}{u}\cdot\frac{b}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ΣD represents the sum of the diameters. This is often a convenient expression to use in testing a proposed arrangement of rods. Thus, using the permissible values of u and v for a 2000-lb concrete, we have:

(*a*) For beams with no web reinforcement and whose maxi-

mum shearing-stress is approximately equal to the allowed value of 40 lbs/in², and $u=80$ lbs/in², we have

$$\Sigma D=\frac{40}{80}\times\frac{b}{\pi}=.16b. \qquad (4)$$

(*b*) For beams fully reinforced and whose shearing-stress is approximately equal to the allowed value of 120 lbs/in², we have

$$\Sigma D=\frac{120}{80}\times\frac{b}{\pi}=.48b. \qquad (5)$$

From these equations, the necessary sum of diameters of rods for bond stress can quickly be estimated, or any proposed arrangement checked.

For example, if a single layer of rods is to be used in case (*a*) the sum of diameters must equal $.16b$, or the necessary average spacing of any size of rod will be not more than $D/.16$ or about $6D$; and for case (*b*) the necessary spacing must not exceed about $2D$ centre to centre. The latter is less than allowable and therefore two layers must be used or else a higher working bond stress.

145. *Use of Anchored Bars.*—Deficient bond strength may, to a limited extent, be supplemented by some form of anchorage, such as nuts and washers, or by means of sharp bends. Such anchorage cannot affect the shearing-stresses or introduce so-called "arch action" to any extent unless an initial tension be applied by means of nuts; it affects chiefly the bond resistance at the end of the bar. Initial slip is not much delayed by bends and hooks, but they do increase very considerably the ultimate bond resistance. As shown in Art. 123 the square bend is not of much value, but if bends are made they should be bends of 180 degrees. In simple beams the ends of the straight bars should be hooked, unless the bond stress is very small.

146. Proportioning of Rectangular Beams.—*General Design.* —The size of a rectangular beam must meet two condi-

tions. To resist the bending moment, we have from Art. 60, eq. (3),

$$bd^2 = \frac{M}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and from Art. 106 the size necessary to resist the shear is

$$bd = \frac{V}{vj}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where v is the allowable maximum shearing stress and V= maximum total vertical (end) shear. Except for very heavy loads and low values of v the requirements for bending moment and for convenient proportions of b and d will give ample cross-section to meet the requirements for shear of eq. (2). Having the value of bd^2 determined from (1), corresponding values of b and d are selected to give convenient and economical proportions. For any given value of bd^2, the value of bd will decrease as d increases. Hence the deeper the beam the less the required cross-section bd. Therefore, until the value of bd becomes as small as determined by (2), the deeper the beam the less the cross-section and the less the amount of concrete. Furthermore, as the *per cent* of steel is a constant, for given values of working stresses, the amount of tension steel is reduced by increasing the depth. When a depth is reached so that bd is equal to the value given by (2) a further increase in depth cannot affect the total cross-section, as bd must be kept constant. In that case the concrete stresses will be decreased and, as shown by the diagram, Pl. II, the percentage of steel can be slightly decreased. These considerations show, therefore, that from the standpoint of the amount of concrete and horizontal tension steel, the deeper the beam the less the material required. On the other hand very deep beams require more steel for shear reinforcement.

In general, then, we may say that for rectangular beams the ratio of depth to breadth should be made as great as consistent with convenience of proportions, space for bars, head room, cost of forms, and similar practical requirements. Gen-

erally speaking the convenient ratio of depth to breadth will vary from a minimum of about $1\frac{1}{2}$ to 1 for small beams up to 3 or 4 to 1 for very large beams. Ample width for placing of bars is an important factor.

147. *Ratio of Length to Depth for Equal Strength in Shear and Moment.*—For any given values of working stresses in steel and concrete and any given arrangement of loading, there is a definite ratio of depth to span length for which the strength in bending is equal to the strength in shear.

For a uniform load we have $M=\frac{1}{8}wl^2$ and $V=\frac{wl}{2}$. Also from (12), Art. 58, $bd^2=\frac{M}{f_s pj}$; and from (2) Art. 146, $bd=\frac{V}{vj}$. Substituting the values of M and V above, and dividing, we find that

$$\frac{l}{d}=\frac{4f_s p}{v}. \qquad (3)$$

For a single concentrated load a similar analysis gives

$$\frac{l}{d}=\frac{2f_s p}{v}. \qquad (4)$$

Taking $f_s=16{,}000$, $p=0.8\%$, and $v=40$ and 120 respectively, we have

For $v=40$, no web reinforcement.
For uniform load, $l/d=12.8$.
For a concentrated load, $l/d=6.4$.
For $v=120$, full web reinforcement.
For uniform load, $l/d=4.2$.
For concentrated load, $l/d=2.1$.

For larger ratios of l/d than the above values the strength in shear will be greater than that in bending. From these ratios it will be easy to judge quickly whether or not the shear is likely to be the determining factor in any case. It is evident that only in relatively deep beams will the shearing strength come into question.

148. Design of Rectangular Beams.—*Problem* 1.—Design a rectangular beam to span 15 ft. c. to c. and to support a uniform load of 5000 lbs/ft, not including weight of beam. The beam is simply supported. Assume a concrete of com-

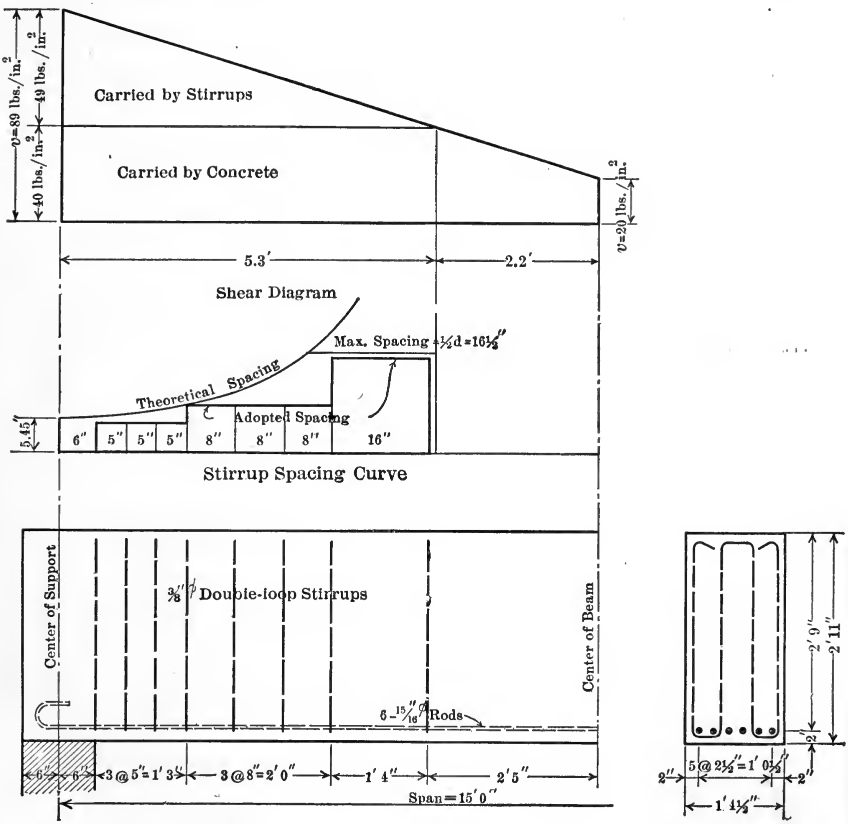


Fig. 3.

pressive strength of 2000 lbs/in² and the unit stresses recommended by the Joint Committee as follows: $f_s=16{,}000$, $f_c=650$, v (for reinforced web) $=120$; $u=80$. The following additional provisions will be adhered to in the design: Allowable stress in stirrups above neutral axis $=10{,}000$ lbs/in²; allowable bond stress in horizontal rods near end, if at least one-third of rods

are bent up, will be increased by 50%, and all rods will be hooked at their ends; where shear reinforcement is used the concrete may be assumed to carry 40 lbs shear per sq. in. throughout, instead of one-third of the shear as recommended by the Joint Committee.

Solution.—See Fig. 3. Assume the weight of the beam at 600 lbs/ft.

Total load $=5000+600=5600$ lbs/ft.

Bending moment $=M=\frac{1}{8}wl^2=\dfrac{5600\times15^2\times12}{8}=1,890,000$ in-lbs.

From Plate II, with $f_s=16,000$ and $f_c=650$, we find $R=107$, and $bd^2=\dfrac{1,890,000}{107}=17,600.$

In this case we will made d equal to about $2b$, whence $d^3=35,200$; $d=32.9$; $b=16.5$.

Adopt values of $d=33$ in. and $b=16.5$. in.

Allowing 2 inches for protective covering below the center of rods gives a total depth of 35 inches. The weight of beam will be $\dfrac{35\times16.5\times150}{144}=600$ lbs/ft. The assumed weight is therefore correct.

The total end shear is

$$V=\frac{5600\times15}{2}=42,000 \text{ lbs.}$$

Maximum unit shear is

$$v=\frac{42,000}{16.5\times.87\times33}=89 \text{ lbs/in}^2,$$

which is allowable with suitable web reinforcement. The dimensions selected will therefore be used.

The steel area can be obtained from the percentage value given in Plate II, of 0.77 or more accurately by direct calculation from the bending moment and the value of j of 0.87 given in Plate II. Thus.

$$A=\frac{1,890,000}{.87\times33\times16,000}=4.12 \text{ sq. in.}$$

To provide this area we may use eight $\frac{13}{16}$-inch rods, giving 4.15 sq. in., or six $\frac{15}{16}$-inch rods, giving 4.14 square inches.

Before deciding upon the arrangement it will be necessary to investigate the bond stress. By eq. (3), Art. 144, using $v=89$ and a value of $u=80$, we have,

$$\Sigma D=\frac{89}{80}\times\frac{16.5}{\pi}=5.85 \text{ inches.}$$

That is, if all rods are straight the sum total of diameters must equal at least 5.85 inches. The eight rods furnish $6\frac{1}{2}$ inches and the six rods 5.6 inches, the latter, therefore, being slightly deficient. This deficiency can readily be met by the use of hooks. If sufficient rods are bent up at the end to take the shear we may increase the allowable value of u by 50% or to 120 lbs/in² (see Art. 133). This gives $\Sigma D=\frac{2}{3}\times 5.85=3.9$ inches. This would require five of the $\frac{13}{16}$-inch rods, or four of the larger rods, leaving three or two respectively to bend up. Inasmuch as this number is inadequate to carry the shear, except for a very short length near the end, all rods will be left straight and extended to the end. The shear will then be taken care of by means of stirrups, which will not be difficult, as the value of 89 lbs/in² is not high. If eight rods are used they will need to be placed in two layers, while the six rods may be placed in one layer. The latter arrangement will be chosen, spacing the rods $2\frac{1}{2}$ inches apart, and 2 inches from the side of the beam. This meets the requirements of Art. 138. If larger rods were used the bond stresses would be excessive.

Shear Reinforcement.—Considering the load a moving load the maximum shear at the centre will be

$$V=\frac{wl}{8}=\frac{5000\times 15}{8}=9360 \text{ lbs.}$$

and the unit shear is

$$v=\frac{V}{bjd}=\frac{9360}{16.5\times 28.7}=20 \text{ lbs/in}^2.$$

The unit shear at the end is 89 lbs/in². The concrete will carry 40 lbs/in², and assuming the maximum shear to vary as a straight line, the value of 40 lbs/in² will occur at a distance from the end equal to $\frac{89-40}{89-20} \times 7.5 = 5.3$ feet. At the support the shear carried by the stirrups is 49 lbs/in². Using a double loop made of $\frac{3}{8}$-inch round rods and a working stress of 10,000 lbs/in, the safe stress in one stirrup is $4 \times 0.11 \times 10{,}000 = 4400$ lbs. The spacing required is from eq. (1), Art. 141, equal to

$$s = \frac{P}{vb} = \frac{4400}{49 \times 16.5} = 5.45 \text{ in.}$$

Between the support and a point 5.3 feet therefrom the spacing will vary inversely as the distance from the latter point. At midway, or 2.65 feet from the end, it will be 10.9 inches, and at the quarter point (4 feet from the end) it will be 22.2 inches. Using no spacing greater than one-half the depth, or 16.5 inches, we adopt the spacing shown in the figure. The bond strength of the stirrups above the beam center will be found ample. The main bars will be bent into hooks at the end to give additional bond strength.

Problem 2.—Design a rectanguiar beam of span length 20 feet, supporting a live load of 2000 lbs. per foot. Use the stresses recommended by the Joint Committee for a concrete of ultimate strength 2000 lbs/in². Additional provisions as in Problem 1.

Solution.—See Fig. 4.

Assume a dead load weight of 550 lbs/ft. Then the bending moment

$$M = \tfrac{1}{8} \times 2550 \times 20^2 \times 12 = 1{,}530{,}000 \text{ in-lbs.}$$

From Plate II, $R = 107$, $j = .87$, $k = .38$ for $f_s = 16{,}000$ and $f_c = 650$,

$$bd^2 = \frac{1{,}530{,}000}{107} = 14{;}300.$$

This requires a beam 30×15.9 inches, 32×14 inches or 34×12.4 inches.

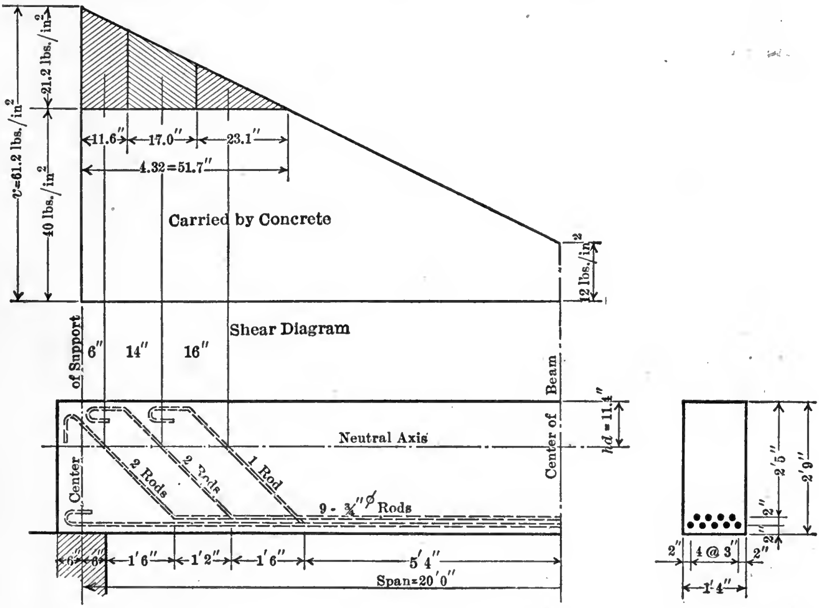


FIG. 4.

We will use a beam 30×16 inches net dimensions. Allowing 3 inches of concrete below the centre of the steel, the weight of the beam becomes

$$33 \times 16 \times \frac{150}{144} = 550 \text{ lbs/ft.}$$

This checks the assumed weight.

$$\text{Area of steel} = \frac{M}{f_s jd} = \frac{1,530,000}{16,000 \times .87 \times 30} = 3.67 \text{ sq. in.}$$

$$\text{Max. end shear} = V = \tfrac{1}{2}wl = \tfrac{1}{2} \times 2550 \times 20 = 25,500 \text{ lbs.}$$

$$\text{Unit end shear} = v = \frac{V}{bjd} = \frac{25,500}{16 \times 0.87 \times 30} = 61.2 \text{ lbs/in}^2.$$

This value of v is such that the beam can be thoroughly reinforced against diagonal tension failure by means of bent-up bars alone.

With a bond stress of 120 lbs/in²

$$\Sigma o=\frac{V}{ujd}=\frac{25{,}500}{120\times .87\times 30}=8.17 \text{ inches.}$$

Knowing A and Σo, the maximum size of rod that may be used can be determined from the relation $D=q\cdot\frac{4A}{\Sigma o}$, where q is the proportional number of rods to remain straight. It will probably be necessary to bend up from $\frac{1}{2}$ to $\frac{2}{3}$ of all the rods for such reinforcement.

Assuming $q=\frac{1}{2}$; maximum dia. $=\frac{1}{2}\times\frac{4\times 3.63}{8.17}=.89$ in.

Assuming $q=\frac{1}{3}$; maximum dia. $=\frac{1}{3}\times\frac{4\times 3.63}{8.17}=.59$ in.

We will try $\frac{5}{8}$, $\frac{11}{16}$, $\frac{3}{4}$, and $\frac{13}{16}$-inch rods.

Twelve rods $\frac{5}{8}$ inch diameter give 3.68 square inches area, 5 rods give 9.8 inches circumference; bend up 7 rods.

Ten rods $\frac{11}{16}$ inch diameter give 3.71 square inches area, 4 rods give 8.64 inches circumference; bend up 6 rods.

Nine rods $\frac{3}{4}$ inch diameter give 3.98 square inches area, 4 rods give 9.44 inches circumference; bend up 5 rods.

Seven rods $\frac{13}{16}$ inch diameter give 3.63 square inches area, 4 rods give 10.22 inches circumferencee; bend up 3 rods.

The choice of rods lies between 9 rods $\frac{3}{4}$ inch diameter and 10 rods $\frac{11}{16}$ inch diameter. The former, 9 rods $\frac{3}{4}$ inch diameter, will be chosen because the size is more common, and the odd number of rods to bend up brings out additional points in the problem.

The 5 rods may be bent up where not required for moment, and since the beam is uniformly loaded, the possible points of bending, considering moment only, are given by eq. (2), Art. 137, $x_n=\frac{l}{\sqrt{A_r}}\sqrt{a_1-(A_r-A_n)+a_2+\ldots+a_n}$. Substituting the proper values, we get for the lengths of the first, third and fifth rods, 3.18 feet, 10.35 feet, and 14.3 feet, respectively.

Shear Reinforcement.—The maximum unit shear at the centre

of the beam is $\frac{1}{8}\times\frac{2,000\times20}{16\times.87\times30}=12.0$ lbs/in². The point beyond which no stirrups are required is

$$\frac{(61.2-40)\times10}{61.2-12}=4.32 \text{ feet from the end.}$$

In Fig. 4 the shear diagram is shown and the shaded portion represents the part to be carried by the reinforcement. Five rods are available for shear reinforcement, and if these are bent up at three points the spacing will be sufficiently close. A single rod will be bent up first and then two groups of two each. They will be bent up so as to make their stresses approximately equal. The total shear (area of shaded triangle $\times$ width) $=\frac{21.2}{2}\times51.7\times16=8750$ lbs. Each rod will then carry $\frac{8750\times.707}{5}=1235$ lbs. We may arrive at the spacing as follows: The spacing at the extreme end (for two rods in a group) would be $s=\frac{1.41\times1235\times2}{21.2\times16}=10.3$ inches. The first rods may then be placed 5 or 6 inches from the end. The next two rods will come about 13 or 14 inches from the first, or, say, 19 inches from the end of the beam. The shear at this point $=21.2\times\frac{52-19}{52}=13.5$ lbs/in². The necessary spacing $=10.3\times\frac{21.2}{13.5}=16$ inches. The second group may therefore be placed $8+6=14$ inches from the first. The length of beam covered by these two groups will then be about $12+16=28$ inches, leaving 24 inches for the last rod, which may be placed, therefore, about at the centre of gravity of this triangle, or $8+8=16$ inches from the second group.

A more exact method, and the one shown in Fig. 4, is to divide the shaded triangle into strips of equal areas and place the rods (or stirrups) at approximately the centres of gravity of the several areas. In this case the area is divided accurately into three parts, the first part (triangle) being one-half as large as each of the others. No great accuracy is necessary,

however, in this calculation, and it will generally be simpler to calculate the necessary spacing at three or four points along the beam and vary the spacing accordingly, using convenient units as in riveting. The lengths of the horizontal portions of the rods are greater than necessary for moment.

149. *Table of Proportionate Spacing of Shear Reinforcement.*—It is sometimes convenient to use a direct method of subdividing a shear triangle such as shown in Fig 4, into a number of equal areas. This can be done by the formula (see Fig. 5)

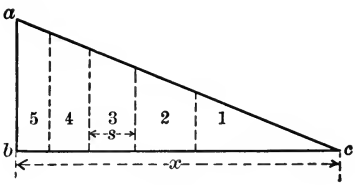


Fig. 5.

$$s=\frac{\sqrt{n'}-\sqrt{n'-1}}{\sqrt{n}}, \quad . \quad . \quad . \quad (1)$$

in which s = width of any strip number n' from c, and n = number of divisions.

Thus, for the third strip where the total number is five,

$$s=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}}x=.143x.$$

A table of coefficients of x is given herewith for any number of divisions up to 10.

Table No. 19.

PROPORTIONATE SPACING OF STIRRUPS.

Proportionate length of strip tributary to each stirrup. To get length in inches, multiply by length of shear triangle in inches.

No. of Divisions of Shear Triangle.	Number of Stirrup from End of Beam.									
	1	2	3	4	5	6	7	8	9	10
1	1.00									
2	.29	.71								
3	.18	.24	.58							
4	.13	.16	.21	.50						
5	.10	.12	.14	.19	.45					
6	.09	.10	.11	.13	.17	.40				
7	.07	.08	.09	.10	.12	.16	.38			
8	.07	.07	.07	.08	.10	.11	.15	.35		
9	.05	.06	.07	.07	.08	.09	.11	.14	.33	
10	.05	.05	.06	.06	.07	.08	.08	.10	.13	.32

Where the reinforcement is assumed to take a certain proportion of the shear, as two-thirds, over that part of the beam where it is used, then the diagram representing shear is as shown in Fig. 6. The centre shear is *cd* and from *c* to *g* the entire shear is carried by the concrete. From *g* to *b* two-thirds is carried by the reinforcement. To use eq. (1) to divide the

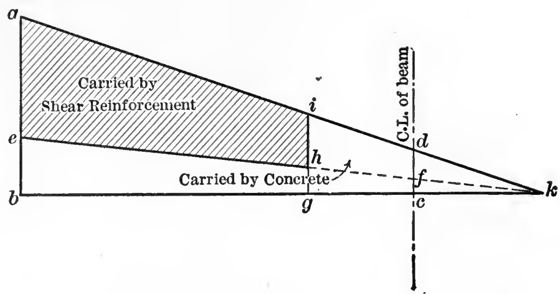


FIG. 6.

area *aehi* into equal parts, use the triangle *eka* in the calculations, and then omit the stirrups from *h* to *k*.

Where a large number of stirrups are required, it will not be desirable to use a different spacing for each one, but to space the stirrups uniformly in groups. This can readily be done from Table 19. Thus, if 12 stirrups are needed and only four different spacings are to be used, say of three stirrups each, then from line 4 of Table 19 we find that the first three must cover a length of $.13x$, the next three $.16x$, then $.21x$. and $.50x$. Or the spacing may be averaged by averaging directly the figures in the table. Thus, for 10 stirrups the spacing of the first four may be made $.055x$, then three at $.08x$, then two at $.115x$, after which the limit of $d/2$ would probably determine the spacing.

150. Proportioning of T-Beams.—*General Design.*—T-beams occur in practice generally where a floor slab and beam are built as a monolithic structure, as in floor construction. Occasionally, also, where heavy girders are required it is expedient to design the beam in the form of a T. Inasmuch as the only

purpose of the concrete below the neutral axis is to bind together the tension and compression flanges, its section is determined by the shearing-stresses involved, and a considerable saving can thus often be effected over the rectangular form. Where the flange is a part of a slab its thickness is determined in the design of the floor, but the width of slab which can be taken as effective flange width must be estimated. A common rule of practice is to count a width of slab not greater than one-fourth of the span length, but this should in fact depend also upon thickness of slab and of the stem of the T. Tests on T-beams with very wide flanges show that the compressive stresses are quite uniformly distributed over the entire width. A still more even distribution of stress is to be expected in a series of T-beams in a continuous floor structure. If, however, the slab or flange is made too wide and thin the shearing stresses along the line aa' and cc', Fig. 7, will be excessive and greater than those along the line $a'c'$. On this account it is desirable to limit the value x to five or six times the thickness of the slab, or three times the width of beam b'. Experiments show that a total flange width of 3 or 4 times the width of the web generally gives ample flange area so that the design of such a T-beam consists mainly in the design of the web or stem, and the proper arrangement of the steel.

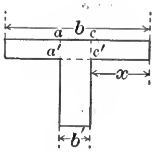


FIG. 7.

Where a T-beam is not connected with a floor system, the size of flange may be selected to meet the conditions at hand. In this case the stem of the beam should first be determined approximately, on the basis of the shearing-stresses to be carried. A suitable flange can then be selected by a few trials, as explained in Art. 82. The deeper the beam the less the amount of steel required for constant cross-section. But T-beams should not be made too deep in proportion to width, as such forms are relatively weak at the junction of stem and flange. All re-entrant angles in rigid material such as concrete are points of weakness and such angles should therefore

be modified by curved lines or by a beveled fillet. A width of beam sufficient to carry the shear and to give plenty of space for the bars is usually ample. The maximum desirable ratio of depth to width may be taken at about two for small beams up to three or four for very large and massive work. Depths are often determined by available head room. Beams of excessive depths are objectionable as being more difficult and troublesome to reinforce properly; the cost of web reinforcement also becomes relatively greater. The flanges should be thoroughly bonded to the web by means of web reinforcement running well up into the flange and, where the flange is wide, by additional cross-reinforcement in the plane of the flange.

151. *Economical Proportions.*—Where a floor-slab forms the flange of a T-beam, then the economical proportions of the stem may be considered. Here the slab forms practically all the compressive area, but does not enter into the cost of the beam. Using the approximate formula, eq. (9) of Art. 79, the area of the steel is equal to $M/f_s(d'+\frac{1}{2}t)$, in which d' is the depth of beam *below* the slab. Then let C = cost of beam per unit length, c = cost of concrete per unit volume, and r = ratio of cost of steel to cost of concrete per unit volume. The total cost per unit length will then be

$$C=c\left[b'd'+\frac{rM}{f_s(d'+\frac{1}{2}t)}\right]. \quad \ldots \ldots \quad (1)$$

From this expression it is evident that the cost will decrease with increased values of f_s under all conditions, and that with a fixed value of $b'd'$ the cost decreases with increase in depth. If d' is fixed then the cost will be a minimum when b' is made as small as possible, and its value will then be determined by the shearing-stress or by the space required for the bars. If the value of b' is assumed as fixed, then there is a definite value of d' which will give minimum cost. Considering d' as variable and b' as constant we find by differentiation that for minimum cost the value of d' is given by the equation

$$d'+t/2=\sqrt{rM/f_sb'}. \quad \ldots \ldots \quad (2)$$

From this expression the best depth for various assumed widths can readily be determined and the desirable proportions finally selected.

152. Design of a T-Beam (Fig. 8).—A beam and slab floor (beams one way only), spans an opening of 24 feet. The

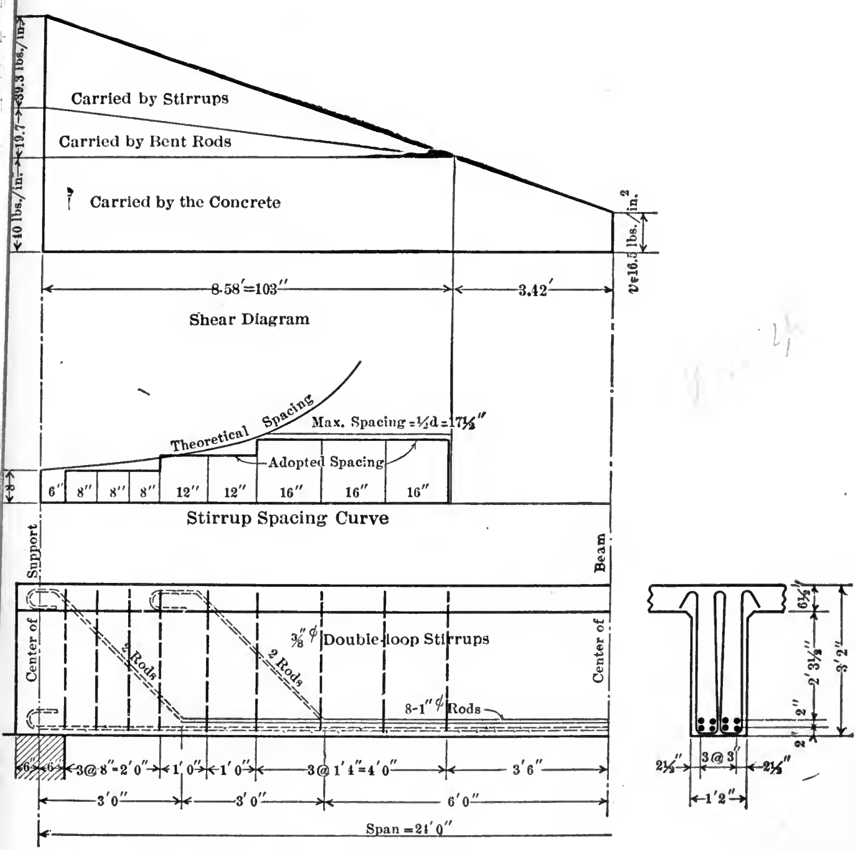


Fig. 8.

slab is 6½ inches thick, the beams are 10 feet on centres, and the live load is 250 lbs/ft². The beams are simply supported, and are to be reinforced against diagonal tension by means

of bent-up bars and stirrups. Assume a concrete of 2000-lb. ultimate strength and working stresses as follows:

$f_s = 16{,}000$ lbs/in² longitudinal steel, 10,000 lbs/in² in stirrups;

$f_c = 650$ lbs/in²;

$v = 40$ lbs/in², where no web reinforcement is used and 100 lbs/in² where web is thoroughly reinforced with bent bars and with stirrups not more than $\frac{1}{2}d$ apart;

$u = 80$ lbs/in² with an allowable increase of 50% as in Art. 148.

Assume the weight of beam below the slab at 460 lbs/ft. Weight of slab carried by the beam

$$= \frac{6\frac{1}{2} \times 12 \times 150}{144} \times 10 = 810 \text{ lbs/ft of beam.}$$

The live load

$$= 10 \times 250 = 2500 \text{ lbs/ft of beam.}$$

The total load

$$= 460 + 810 + 2500 = 3770 \text{ lbs/ft.}$$

Assume $j = .93$, then $V = 12 \times 3770 = 45{,}200$ lbs, and

$$b'd = \frac{V}{vj} = \frac{45{,}200}{100 \times .93} = 487 \text{ sq. in.}$$

A beam of $14'' \times 35''$ gives a web area of 490 in². The ratio of d to b' is $2\frac{1}{2}$, and the 14-inch width will hold approximately 4 rods in a single row. Allowing 3 inches for protective covering on the steel for two rows of rods, the weight of beam below the slab

$$= (35 + 3 - 6\tfrac{1}{2}) \times 14 \times \frac{150}{144} = 460 \text{ lbs/ft.}$$

The assumed weight is therefore correct.

The maximum bending moment

$$= \tfrac{1}{8} \times 3770 \times 24^2 \times 12 = 3{,}260{,}000 \text{ in-lbs.}$$

The economical depth for moment, according to Art. 151, depends upon the relative cost of concrete and steel, the unit stress f_s and the bending moment. With steel at 3 cents per pound and concrete at $8.00 per yard in place, the value of r becomes $\frac{.03\times490}{8.00/27}=50$, and if $b'=14$ inches, then

$$d'+\frac{t}{2}=\sqrt{\frac{rM}{f_sb'}}=\sqrt{\frac{50\times3{,}260{,}000}{16{,}000\times14}}=26.9 \text{ inches, and}$$

$$d=26.9+\frac{6\frac{1}{2}}{2}=30.15 \text{ inches.}$$

The economical depth is less than the depth required for shear, assuming a 14-inch breadth, and hence the 35-inch depth obtained above must be used. Since $b'd$ remains fixed (the stem of the beam is designed for shear), no economy results in making the beam wider and shallower.*

The value of f_c must now be investigated. The effective width of flange is limited to one-fourth the span length and to $12t+b'$. One-fourth span length $=72$ inches, and $12t+b'$ $=92$ inches.

From Plate VII, with $\frac{t}{d}=\frac{6\frac{1}{2}}{35}=.185$, and $R=\frac{3{,}260{,}000}{72\times35^2}=37$, the unit compressive stress in the concrete is found to be 350 lbs/in², and j is about 0.93. This checks the assumed value of j.

---

* If the live load were concentrated at the centre, the resulting moment would be

$$\tfrac{1}{8}\times1270\times\overline{24}^2\times12+\frac{2500\times24\times24}{4}\times12=5{,}415{,}000 \text{ in-lbs.}$$

A beam with 35″×14″ web would be required for shear as before and the economical depth for moment would be

$$d=d'+t=\sqrt{\frac{rM}{f_sb'}}+\frac{t}{2}=\sqrt{\frac{50\times5{,}415{,}000}{16{,}000\times14}}+\frac{6.5}{2}=34.7+3\tfrac{1}{4}=37.95 \text{ inches.}$$

Therefore, assuming a 14-inch width as necessary to hold the steel, it would be economical to change the depth from 35 inches to 38 inches, making the maximum shearing stress less than 100 lbs per in².

The steel area required is $A=\frac{M}{f_s jd}=\frac{3,260,000}{16,000\times.93\times35}=6.26$ in². The circumference of the steel must be at least $\Sigma o=\frac{V}{ujd}$ $=\frac{45,200}{120\times.93\times35}=11.55$ inches, but should preferably be about twice as much to permit bending up of rods.

Eight rods 1 inch diameter provide 6.28 square inches of area, and 25.13 inches of circumference Four straight rods will provide $4\times3.14=12.56$ inches circumference; hence four rods may be bent up for web reinforcement where no longer needed to resist bending moment. The points where the rods may be bent up can be determined from a moment diagram, or by the use of Art. 137. From this we get

$$x_2=12.0 \text{ feet}=\text{length of second rod},$$

$$x_4=16.95 \text{ feet}=\text{length of fourth rod}.$$

The first pair of rods will be bent up at 6 feet, and the second pair at 9 feet from the centre of the beam.

The maximum shearing-stress at end

$$=\frac{45,200}{14\times.93\times35}=99 \text{ lbs/in}^2.$$

Maximum shearing-stress at centre

$$=\frac{2500\times24}{8\times14\times.93\times35}=16.5 \text{ lbs/in}^2.$$

Since it simplifies the work and introduces no large error, it will be assumed that the unit shearing-stress varies uniformly from 16.5 at the centre to 99 at the end, as shown in Fig. 8. The concrete will carry 40 lbs/in² without web reinforcement; the remainder must be taken care of by bent bars and stirrups. No web reinforcement is necessary at a distance from the support of

$$\frac{(99-40)12}{(99-16.5)}=8.58 \text{ feet}=103 \text{ inches}.$$

Since only two pairs of rods are available for bending, they cannot serve to completely reinforce the beam except for a very short distance. Instead of attempting to do this, however, the bars will be bent up at intervals of about the depth of the beam and assumed to carry about one-third of the shear not carried by the concrete. This will give a maximum shearing-stress for concrete and bent bars combined of 60 lbs/in², which is a reasonable value. It will be noted also that in a large beam of this kind it is not convenient to arrange bent-rod reinforcement sufficiently near the end to be fully effective. Stirrups should therefore be generally used near the end, even though bent rods may be sufficient elsewhere. In the case of continuous beams, bent rods can be made very effective very near the support (see Art. 153).

At the support, the shear to be taken care of by stirrups will be $\frac{2}{3}(99-40)=39.3$ lbs/in², and the required spacing of $\frac{3}{8}$-inch round double loop stirrups for $f_s=10{,}000$ lbs/in² will be

$$s=\frac{P}{vb'}=\frac{4\times0.11\times10{,}000}{39.3\times14}=8.0 \text{ inches.}$$

At $\frac{1}{2}\times103$ inches from the support, v is half as great, and s becomes 16 inches; at $\frac{3}{4}\times103$ inches from the support v is one-fourth as great and s becomes $4\times8=32$ inches. Fig. 8 shows graphically the variation in stirrup spacing, and from the curve the desired spacing may be determined. The maximum spacing of vertical stirrups is limited to $d/2$ or $17\frac{1}{2}$ inches. The bent-up rods will be turned up at 45° as near the ends of the beam as practicable, because the shear is greatest near the support, and because at points nearer the centre the stirrups are usually spaced closer together than theoretically required. Sufficient length of bar and stirrup must be provided above the neutral axis to give the necessary bond strength. For $f_s=10{,}000$, the length of imbedment for bond is $\dfrac{10{,}000}{4\times80}=31$ diameters. For the stirrups this would require a length of about $11\frac{1}{2}$ inches. This is amply secured by the loops and hooks. The actual

stresses in the bent-up rods will now be estimated. The shear per linear inch carried by these bent-up rods is $\frac{1}{3}\times 59\times 14=276$ lbs/in at the end of the beam and zero at a point 103 inches from the end. The total shear carried is then $\frac{103\times 276}{2}=14{,}200$ lbs. Dividing this equally between the four rods the stress in each one is $\frac{14{,}200 \cos 45°}{4\times .785}=3190$ lbs/in², which is much less than the rods can safely carry. Their bond strength is also ample.

End bearing.—Using a unit bearing pressure of 400 lbs/in² the area of support must be $\frac{45{,}200}{400}=113$ square inches. Applying a factor of $1\frac{1}{2}$ for unequal pressures, the length of bearing becomes $\frac{113\times 1.5}{14}=12$ inches.

153. Design of Continuous Beams.—In steel construction, when several successive beam spans are constructed, resting on a series of columns, each is usually designed as a simple beam, the end details being unsuited to carry the negative bending moments which would result from continuity of action. In reinforced concrete, on the other hand, it is usually desirable to construct the beams as a continuous or monolithic piece of work rather than as disconnected, simply-supported beams. This method of design has been a gradual development. At first the structure was merely tied together by overlapping some of the reinforcing bars at the support; then to prevent cracks on the upper surface over or near the support, some reinforcement was placed in the upper part of the beam. These conditions lead to the design and construction of such beams as true continuous girders, proportioning the reinforcement over the supports to carry the negative moments involved.

Fig. 9 represents the general variation in moment and shear in a continuous beam uniformly loaded. The negative moment is a maximum over the support, and is larger than the positive moment between supports. It decreases rapidly

from the support. The point of inflexion is about .2l from the support. The shear is about the same as in a simple beam; it changes sign suddenly at the supports.

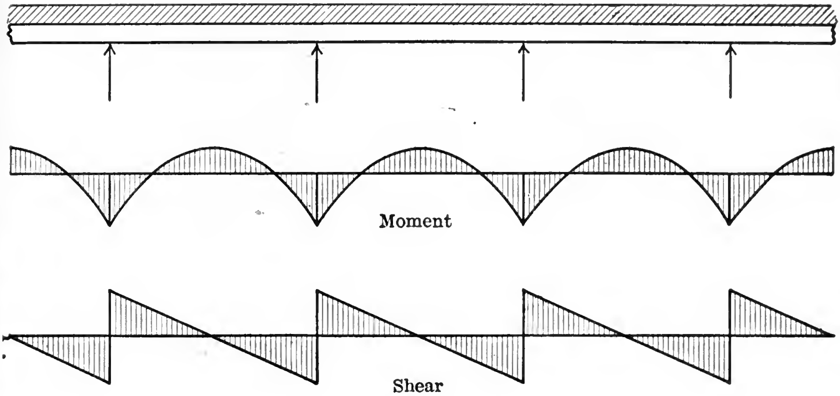


FIG. 9.

154. *Arrangement of Reinforcement.*—Various arrangements of reinforcement may be used. Generally, some of the lower rods are bent up and extended over the support to furnish part of the upper reinforcement. Some of the lower reinforcement should extend well into the column and preferably overlap so as to bind the structure thoroughly together against contraction cracks. Fig. 10 illustrates a common arrangement of reinforcement for a continuous beam. A portion of the lower rods are extended straight through, and a portion are bent up and extended across the top to furnish part of the negative reinforcement. Fig. (*b*) shows the scheme of arrangement in detail. Additional short top rods are often used to make up the necessary reinforcement for negative moment, and may be bent down at their ends to furnish additional shear reinforcement. In this type of beam the maximum moment occurs at the same point as the maximum shear which gives conditions more unfavorable as regards diagonal cracks than in the case of the simple beam. Special attention must therefore be given to the shear reinforcement.

Diagonal tension cracks tend to form first near the support, the shear being a maximum at this place; and they tend to open up at the top of the beam, this being the tension face. Fig. 12 illustrates the general form of such cracks in this case. It is important to take due account of these conditions in the arrangement of web reinforcement. Stirrups should be looped about or attached to the upper rods as these are the tension rods. It is not sufficient merely to anchor the ends of the

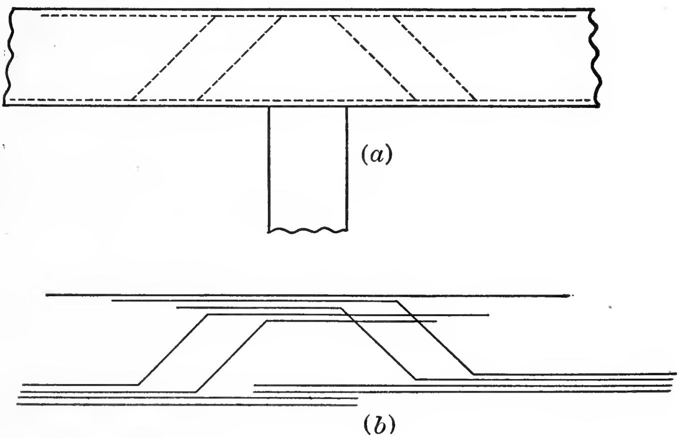


Fig. 10.

stirrups in the concrete; they should be looped about the rods themselves or firmly attached thereto. If the ordinary single stirrup bent in U form is used, it should be inverted, the free ends being well anchored to the concrete below the neutral axis. Inadequate shear reinforcement in continuous beams is not uncommon, due mainly to the overlooking of the necessary relation of such reinforcement to the tension side of the beam.

155. *Bond Stress.*—As the bond stress is a linear function of the shear, it follows that it changes sign suddenly at the support. This condition gives rise to a sudden change in the *direction* of bond stress. Thus, in Fig. 11, on the left of the support, the concrete pulls towards the left on the upper

rod, and on the right it pulls towards the right, as shown by the small arrows. Any slipping increases the deformation of the concrete at once, and hence increases the tension in the concrete at the centre. Likewise, at the bottom, any slip tends to increase the compressive stress in the concrete. It follows, therefore, that where rods continue over the support in continuous beams, the bond stress should be well provided for each side of the centre, otherwise the deformations and stresses in the concrete will be excessive. As a matter of fact, the exact theoretical conditions can hardly be realized, and the variation in bond stress probably follows more nearly a rounded curve such as shown by the dotted line in Fig. 11. It is also true that the bond stress is seldom so well taken care of that there is not some slight cracking immediately over the support.

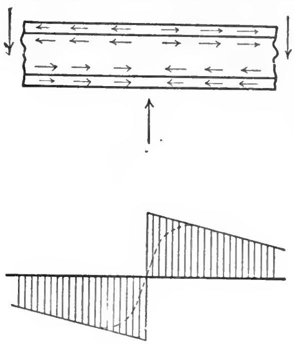
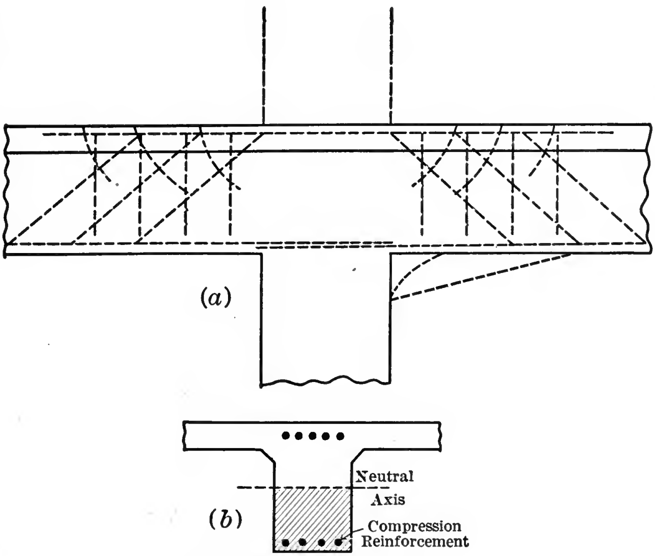


FIG. 11.

156. *T-beams as Continuous Beams.*—In floor construction a T-beam designed for positive moments becomes a rectangular beam over the supports where the moments are negative (see Fig. 12). The tension side is uppermost, and, neglecting the tension in the concrete, the flange of the T is of no value in resisting this negative moment. From the moment diagram of Fig. 9 it is seen that the negative moments are larger than the positive moments, and hence it will generally be found that a continuous T-beam designed for positive moments will furnish inadequate compressive area at the support for the negative moment. This condition is met by the use of compressive reinforcement, and, generally, to some extent, by allowing a little greater unit stress at this point then elsewhere. Fig. 9 shows that the negative moment falls off very rapidly from the theoretical maximum over the support so that within a few inches from the support the moment will be much below the

maximum. In view of this condition, the allowable unit stress may reasonably be increased slightly. An increase of 15% is



SECTION NEAR COLUMN

FIG. 12

recommended by the Joint Committee. Increasing the depth of the beam near the support as indicated in Fig. 12 (*a*) is also a common method of providing for the negative moment.

CHAPTER VI.

DEFLECTION OF BEAMS.

157. Deflection formulas for homogeneous beams can be interpreted semi-rationally to make them applicable to reinforced concrete beams. So interpreted, they yield results in fair agreement with actual measured deflections.

158. **General Theory.**—As is well known, a concrete-steel beam under full working load contains one or more cracks at or near the section of maximum bending moment or else the condition there is near the cracking stage, and to compute the maximum unit fibre stresses at such section, engineers rightly assume the presence of a tension crack, and, in effect, that it has extended to the neutral axis. Since the deflection depends on the stress at all sections, and the cracked sections are comparatively very few, a deflection formula should be based on the intact section. It may be thought that a cracked section influences the deflection more than an intact one, the idea is correct, but the effect of incipient cracking on the deflection is not as great as on the fibre stress at the section. These effects are entirely different in "order of magnitude," the first is not noticeable at all in careful measurements on deflections due to increasing loads, whereas the latter certainly would be if fair measurements of fibre stress *at* a section of a beam were possible. To simplify certain relations, it will be assumed that the depth of the intact section for use in the deflection formula extends from the top of the beam to the centre of the steel, this in effect assumes all sections cracked from the bottom to the centre of the steel.

Deflection formulas for homogeneous beams imply that

the material of the beam obeys Hooke's law ("stress is proportional to strain"), up to working stresses at least, and that the moduli of elasticity of the material for tension and compression are equal. While it is true that concrete does not obey the law strictly, still its stress-strain relation for compression is nearly linear up to working stresses. But the stress-strain relation for tension is far from linear, and the assumption that it is, herein made for simplicity in formulas, must be regarded as a rough approximation. It is true that the "initial moduli" (Art. 24) of concrete for compression and tension are nearly equal, but the deflection of a beam depends on the elongations and shortenings of all the fibres, and hence not upon initial modulus but on some sort of a mean value. This is not the modulus corresponding to the mean unit fibre stress, but certainly the average or secant modulus is nearer correct than the initial or the modulus at the maximum unit stress.

The formulas also imply that the moments of inertia of the cross-sections of the beam are equal. This condition is not fulfilled in most reinforced concrete-beams, due account being taken of the steel, because of presence of bent-up rods and stirrups. Still the amount of steel in, and hence the moments of inertia of, sections in the middle third or middle half are commonly constant; and since the middle half contributes nearly 85% of the maximum deflection in the case of a simple beam constant in section and uniformly loaded, and 82% when the beam is loaded at the two outer points, it must be that a small change in the moments of inertia of end sections of a simple beam would produce a much smaller change in the maximum deflection. In fact, if a simple beam is uniformly loaded, for example, and the moment of inertia of sections in its middle half is I_1, and that of sections in its outer quarters is I_2, then its maximum deflection is $Wl^3(67I_2+13I_1)/6144EI_1I_2$; and if the sections are uniform and the common moment of inertia is I_1, then the maximum deflection is $5Wl^3/EI_1\,384$; hence the ratio of the deflections is

$(67I_2+13I_1)/80I_2$, and if I_1 and I_2 differ by 10%, say, the maximum deflections differ by less than 2%.

For the reasons stated above, the deflection formulas for homogeneous beams will be used for reinforced-concrete beams, but modified in accordance with the following assumptions:

1. That the representative or mean section has a depth equal to the distance from the top of the beam to the centre of the steel;
2. That it sustains tension as well as compression, both following the linear law;
3. That the proper mean modulus of elasticity of the concrete equals the average or secant modulus up to the working compressive stress; and
4. That the allowance for steel in computing the moment of inertia of the mean section should be based on the amount of steel in the mid-sections.

159. Deflection of Rectangular Beams.—For homogeneous beams the deflection formulas commonly involve the load or the maximum unit fibre stress. The following are corresponding formulas for rectangular reinforced concrete beams:

$$D=\frac{c_1}{E_s}\frac{Wl^3}{bd^3}\frac{n}{\alpha}, \qquad (1)$$

and

$$D=\frac{1}{2}\frac{c_1}{c_2E_s}\frac{f_c l^2}{d}(kj)\frac{n}{\alpha}, \qquad (2)$$

or

$$D=\frac{c_1}{c_2E_s}\frac{f_s l^2}{d}(pj)\frac{n}{\alpha}. \qquad (3)$$

In these the notation is as follows:

D=maximum deflection (if desired in inches, the units specified below should be used);
b=breadth of the section (in.);
d=depth of the section to the centre of the steel (in.);

c_1 = the numerical coefficient in the formula for deflection of homogeneous beams, c_1Wl^3/EI, depending on the loading and support (see Table No. 20).

c_2 = the numerical coefficient in the formula for maximum bending moment, c_2Wl, also depending on the loading and support (see Table No. 20).

E_s = modulus of elasticity of the reinforcing steel (lbs/in²);

E = modulus of elasticity of the concrete (lbs/in²);

n = ratio of the moduli of elasticity of steel and concrete;

p = steel ratio (area of steel section $\div bd$);

α = a numerical coefficient depending on p and n;

f_c = greatest unit compressive stress in the concrete (lbs/in²);

f_s = greatest unit tensile stress in the steel (lbs/in²);

k = proportionate depth of the neutral axis;

j = proportionate distance of the centroid of the compressive stress from the steel.

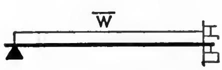
Table No. 20 gives values of c_1/E_s for use in formula (1) and values of c_1/c_2E_s for formulas (2) and (3) for certain standard cases more or less close approximations to which are met in practice; and the diagrams (Figs. 1 and 2) furnish values of n/α, kj, and pj. It is recommended that 8 or 10 be used for n in the first diagram (see Art. 163); in the second that value of n is to be used which the computer prefers in his own formulas, tables, or diagrams for the strength of beams. These two values of n will probably be unlike; the apparent inconsistency is discussed at the close of this article.

Example 1.—A concrete beam rests on end supports 16 feet apart, the breadth of its section is 10 inches, the depth (to the steel) is 15 inches, the reinforcement consists of four $\frac{3}{4}$-inch rods extending along the whole length (and stirrups). What is its probable deflection when sustaining a uniform load of 10,000 lbs., including its own weight?

The amount of steel is 1.767 in², hence $p = 1.767 \div 150 = .012$. Entering the diagram (Fig. 1) at percentage 1.2, tracing upward to the $n=8$ curve say, and then horizontally, it is found that $n/\alpha = 76$.

Table No. 20.

SCHEDULE OF COEFFICIENTS.

$D=c_1Wl^2/El$ $M=c_2Wl$	c_1	c_2	In millionths. $\frac{c_1}{*E_s}$	In millionths. $\frac{c_1}{*c_2E_s}$
	$\frac{1}{3}$	-1	.0111	.0111
	$\frac{1}{8}$	$-\frac{1}{2}$	.00417	.00834
	$\frac{1}{48}$	$\frac{1}{4}$	.000694	.00278
	$\frac{5}{384}$	$\frac{1}{8}$	.000434	.00347
	.00932	$-\frac{3}{16}$	.000301	.00160
	.0054	$-\frac{1}{8}$	.000180	.00144
	$\frac{1}{192}$	$\pm\frac{1}{8}$	.000173	.00139
	$\frac{1}{384}$	$-\frac{1}{12}$	.000087	.00210

* For $E_s=30{,}000{,}000$ lbs/in² in this schedule.

From Table No. 20, it is found that $c_1/E_s=0.000434/1{,}000{,}000$, hence from eq. (1)

$$D=\frac{.000434\times 10{,}000\times 192^3\times 76}{1{,}000{,}000\times 10\times 15^3}=.07 \text{ in.}$$

Example 2.—The deflection of the beam described in the preceding example is desired, (1) when it is loaded so that the working compres-

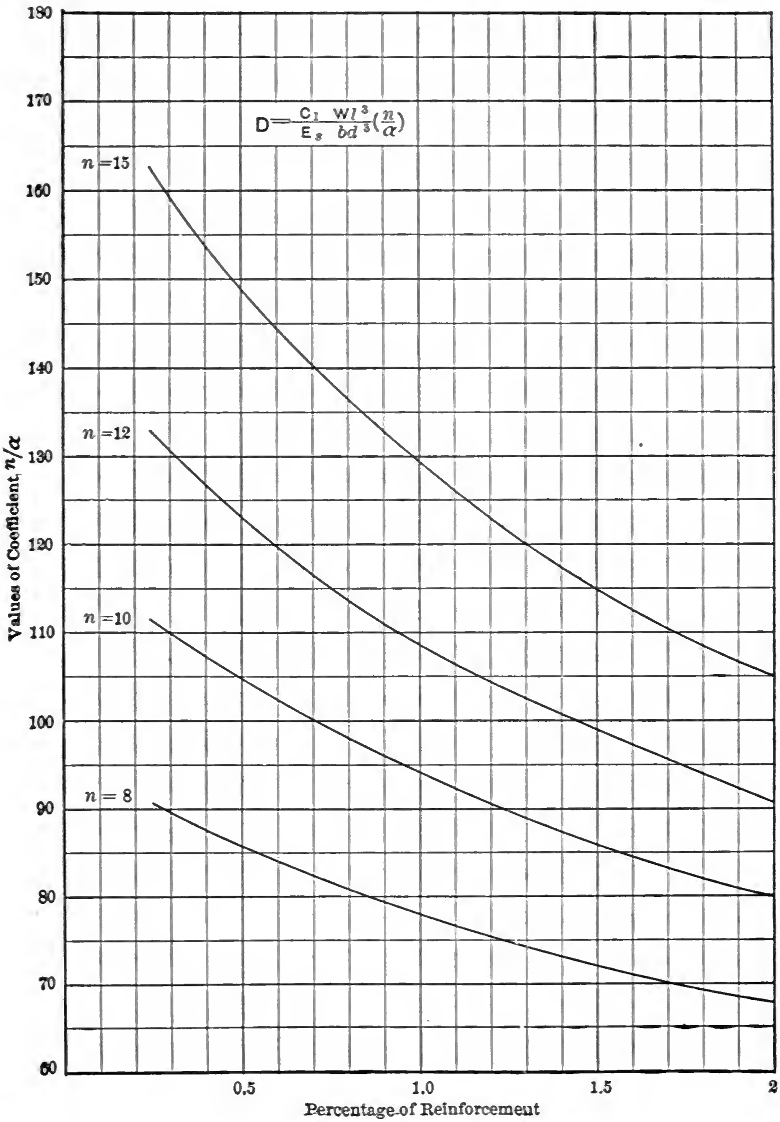
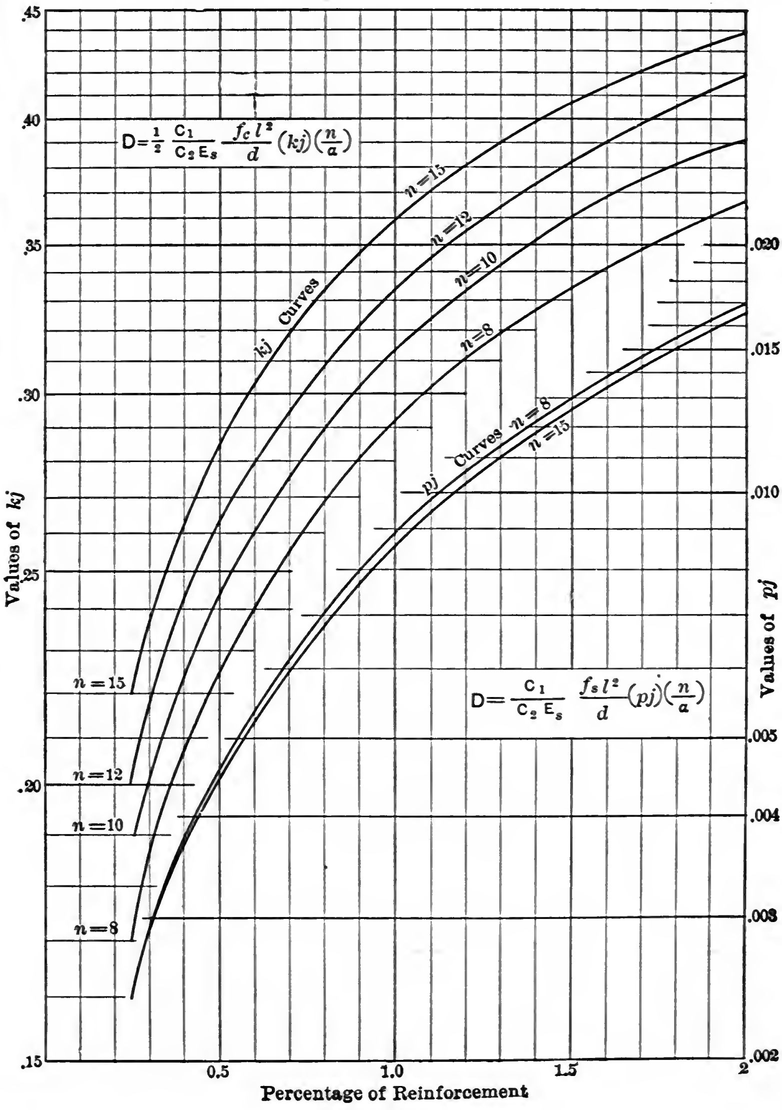

$D = \frac{C_1}{E_s} \frac{Wl^3}{bd^3} \left(\frac{n}{\alpha}\right)$
n = 15
n = 12
n = 10
n = 8
180
170
160
150
140
130
120
110
100
90
80
70
60
Values of Coefficient, n/α
0.5
1.0
1.5
2
Percentage of Reinforcement

FIG. 1.



$D=\frac{1}{2}\frac{C_1}{C_2 E_s}\frac{f_c l^2}{d}(kj)\left(\frac{n}{a}\right)$
$D=\frac{C_1}{C_2 E_s}\frac{f_s l^2}{d}(pj)\left(\frac{n}{a}\right)$
kj Curves
pj Curves
n=15
n=12
n=10
n=8
Values of kj
Values of pj
.45
.40
.35
.30
.25
.20
.15
.020
.015
.010
.005
.004
.003
.002
0.5
1.0
1.5
2
Percentage of Reinforcement

FIG. 2.

sive fibre stress is 500 lbs/in², and (2) when the working stress in the steel is 14,000 lbs/in².

(1) From the schedule it is seen that $c_1/c_2E_s = .00347/1{,}000{,}000$, and, as in example 1, $n/\alpha = 76$. Entering the diagram (Fig. 2) at $p = 1.2\%$ and tracing upwards to the $n = 15kj$ curve (a value of n much used in strength formulas), and then horizontally to the left, we find that kj is .38; hence from eq. (2),

$$D = \frac{.00347 \times 500 \times 192^2 \times .38 \times 76}{2 \times 1{,}000{,}000 \times 15} = .06 \text{ in.}$$

(2) Entering the diagram at $p = 1.2\%$ and tracing upward to the $n = 15pj$ curve and then horizontally to the right we find that $pj = .0102$; hence from eq. (3)

$$D = \frac{.00347 \times 14{,}000 \times 192^2 \times .0102 \times 76}{1{,}000{,}000 \times 15} = .09 \text{ in.}$$

Analysis for Formulas and Diagrams.—Since the total tension (in concrete and steel) and the total compression are equal (see Fig. 3), at any section,

$$\frac{1}{2}\frac{f_s}{n}b(d-kd) + f_sA = \tfrac{1}{2}f_cbkd.$$

Also $f_s/n = f_c(1-k)/k$ and $A = pbd$, and these values substituted in the first equation yield one from which it follows that

$$k = \frac{1+2np}{2+2np}. \quad \ldots \ldots \ldots \quad (4)$$

The moment of inertia, with respect to the neutral axis, of the part of section in compression is $\frac{1}{3}bk^3d^3$, that of the concrete section in tension is $\frac{1}{3}b(1-k)^3d^3$, and that of the weighted steel sections is practically $nA(1-k)^2d^2$; hence

$$I = \tfrac{1}{3}[k^3 + (1-k)^3 + 3np(1-k)^2]bd^3,$$

or

$$\alpha = \tfrac{1}{3}[k^3 + (1-k)^3 + 3np(1-k)^2], \quad \ldots \ldots \quad (5)$$

and

$$D = c_1 Wl^3 / E_c I = c_1 Wl^3 n / E_s bd^3 \alpha,$$

which is eq. (1).

From eqs. (4) and (5), the value of α for any values of p

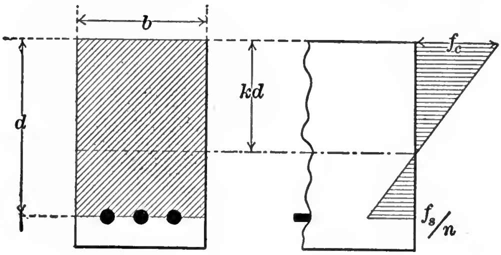


FIG. 3.

and n may be computed; a sufficient number of these were thus computed to determine the n/α curves in Fig. 1.

The transformation of the deflection formula (1) (in terms of the load) into (2) and (3) (in terms of the working unit stresses f_c and f_s respectively) will now be made. For this purpose, strength formulas based on cracked sections (no tension in concrete) and a linear variation of compression are used. These well-known strength formulas based on concrete and steel are respectively, $M = \frac{1}{2} f_c kjbd^2$ and $M = f_s pjbd^2$. Since $M = c_2 Wl$ also, $W = \frac{1}{2} f_c kjbd^2 / c_2 l = f_s pjbd^2 / c_2 l$. These two values of W substituted in eq. (1) yield eqs. (2) and (3) respectively.

The formulas for k and j of Art. 55 are also well known; they are

$$k = \sqrt{2pn + (pn)^2} - pn, \quad \ldots \quad (6)$$

and

$$j = 1 - \tfrac{1}{3} k. \quad \ldots \quad (7)$$

By means of these the values of kj and pj can be computed for any values of p and n; a sufficient number of these were thus computed to determine the kj and pj curves of Fig. 2.

Choice of different values of n in n/α and kj or pj for use

in any particular case is not an inconsistency. The first value depends on the unit fibre stresses at all points of the beam, and when the numerical value is chosen from experiments on deflection, then n becomes also a sort of empirical coefficient-

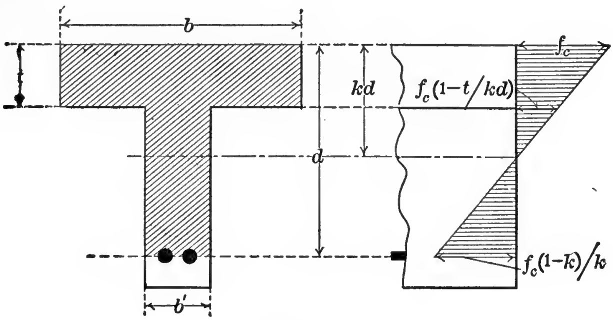


FIG. 4.

making correction for various errors in the deduction of the deflection formula, whereas the second depends on the unit stresses in the cracked section and when its numerical value is chosen from experiments on the strength of beams, then it also becomes in part an empirical coefficient correcting errors of approximation in the strength formulas used.

160. Deflection of T-Beams.—Under the four assumptions stated in Art. 158, the deflection formula for T-beams in terms of the load becomes

$$D=\frac{c_1}{E_s}\frac{Wl^3}{bd^3}\frac{n}{\beta}, \qquad (1)$$

in which β is a coefficient depending upon the steel ratio and n, b width of flange, d depth to steel; other symbols are explained in Art. 159.

In accordance with assumption 2, the neutral axis of the representative section will be in the web, or stem, generally, as is implied in Fig. 4. Then the total tension and compression at the section are given by

$$T=b'(1-k)d\tfrac{1}{2}f_c(1-k)/k+pbdnf_c(1-k)/k$$

and $$C=bt\tfrac{1}{2}[f_c+f_c(1-t/kd)]+b'(kd-t)\tfrac{1}{2}f_c(1-t/kd).$$

Since $T=C$, their values may be equated; the resulting equation leads to

$$k=\frac{np+\frac{1}{2}\left[\frac{b'}{b}-\frac{b'}{b}\left(\frac{t}{d}\right)^2+\left(\frac{t}{d}\right)^2\right]}{np+\frac{b'}{b}-\frac{b'}{b}\frac{t}{d}+\frac{t}{d}}. \quad . \quad . \quad . \quad . \quad (2)$$

The moment of inertia of the concrete-steel section, the steel area being weighted n-fold, is given by

$$I=bd^3\left[k^3-\left(1-\frac{b'}{b}\right)\left(k-\frac{t}{d}\right)^3+\frac{b'}{b}(1-k)^3+3pn(1-k)^2\right]\frac{1}{3},$$

and if β be used to denote this coefficient of bd^3, then

$$\beta=\frac{1}{3}\left[k^3-\left(1-\frac{b'}{b}\right)\left(k-\frac{t}{d}\right)^3+\frac{b'}{b}(1-k)^3+3pn(1-k)^2\right]. \quad (3)$$

Example.—A T-beam rests on end supports 10 ft. apart and sustains loads of 5000 lbs. at its third points. The dimensions of the section are $b=16$ in., $b'=8$ in., $d=10$ in., and $t=3\frac{1}{4}$ in.; and the reinforcement consists of three $\frac{3}{4}$-in. square bars. What is the probable deflection due to the load?

Solution. The steel ratio is .011; and with $n=8$, eq. (2) gives $k=.485$, and eq. (3) gives $\beta=.0835$. Now for loads at third points, $c_1=23/1296$; hence

$$D=\frac{23}{1296}\,\frac{10{,}000\times120^3}{30{,}000{,}000\times16\times10^3}\,\frac{8}{0.0835}=.06 \text{ in.}$$

161. Experiments on Deflections of Beams.—Probably the most complete and accurate deflection measurements ever made are those by Bach. About 50 rectangular and 20 T-beams are reported on by him.* The *rectangular beams* were 2 m. long, 30 cm. deep and 15, 20, or 30 cm. wide. They were

* Mitteilungen über Forschungsarbeiten auf Gebiete des Ingenieurwesens, Hefte 39, 45, 46, 47 (1907).

reinforced with a single straight rod, several straight rods, straight rods with stirrups, or several rods, some bent up; the percentages varied from about .4 to 1.35. The T-beams were 3 m. long, 45 cm. wide, 48 cm. deep, flange 10 and web 20 cm. thick. They were reinforced with straight rods, with or without stirrups, or rods, some bent up, with and without stirrups; the percentage of steel was about .8 in all of them. The beams rested on end supports, and were loaded at the quarter or third points. They were made in sets of three (as nearly alike as possible) and the load-deflection curves for any set are in remarkably good agreement. Deflections were measured at 5 or 7 different points along the beam and to the nearest .005 mm.

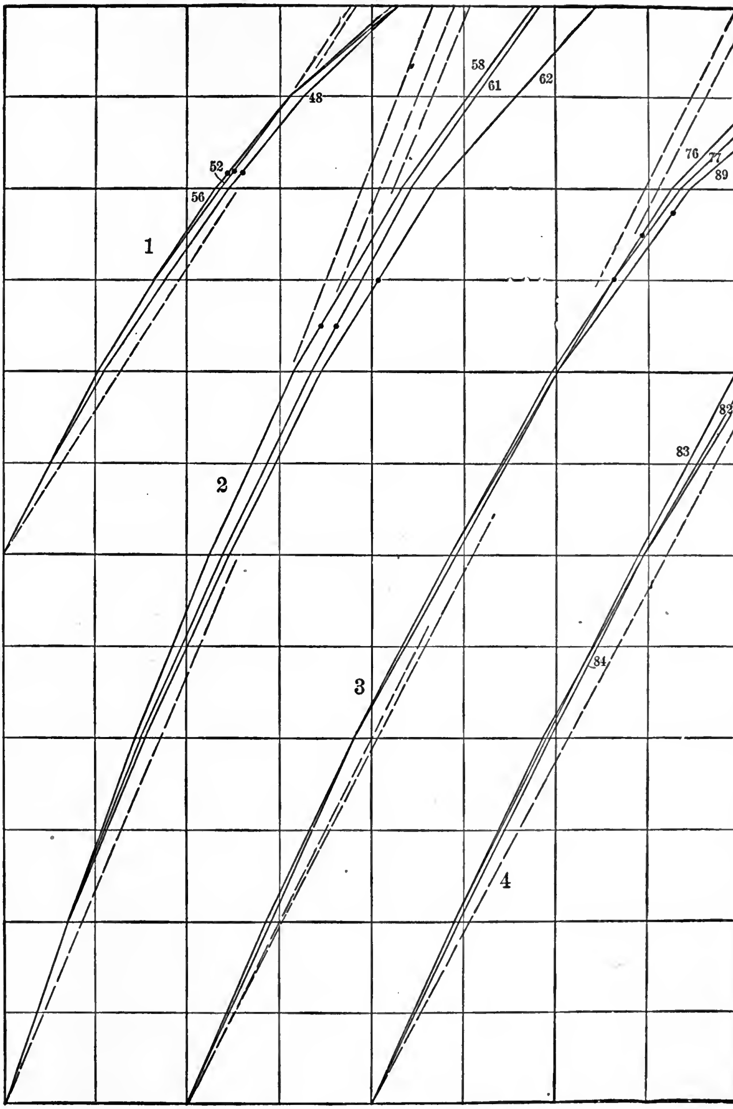
Fig. 5 shows the load-deflection curves for 4 sets of beams. Groups 1 and 2 relate to rectangular beams; in (1) the beams were 15 cm. wide, reinforced with .5% of steel in 3 rods, 2 bent up at each end; in (2) the beams were 20 cm. wide and reinforced with 1.35% of steel in 3 rods, 2 bent up at each end. Groups 3 and 4 relate to T-beams; in the beams of (3) there were 3 straight rods (.8%) and 24 stirrups, and in those of (4) there were 5 rods, 4 bent up at each end, (.87%) and 24 stirrups. Only a part of each curve is given. The dot on each corresponds to one-quarter ultimate load; dots on extensions of group 4 would be a trifle higher than in group 3.

The dashed lines are graphs of the deflection formulas (see Arts. 159 and 160) corresponding to the various beams, n having been taken as equal to 8 for reasons given in the next article. The deflection formula agrees as well with other sets in Bach's tests except in a few cases in which the reinforcement consisted of a single straight rod and stirrups.

Deflection measurements on beams tested in America seem not to have been made with special care, as there is considerable discordance in the published results. Among the best are some reported by Talbot, a few * of which are represented

* Bulletin Univ. of Ill. Eng. Exp. Station, No. 12 (1907); Eng. News, Vol. LX, p. 145 (1908).

Loads.- Scales: for 1 and 2, 1 in.=1000 kg; for 3 and 4, 1 in.= 2000 kg.



Deflections.- Scale: 1 in.=0.2 mm.

FIG. 5.

in Fig. 6. Groups 1 and 2 relate to two sets of T-beams, 12 in. deep (over all), flange $3\frac{1}{4}$ and web 8 in. thick; the span was 10 ft., and loads at third points. The three in group 1 were 16 in. wide, reinforced with straight bars (about 1%) and stirrups, the three in group 2 were 24 in. wide, reinforced as others except some rods were bent up. Curve 3 is for a very large rectangular beam; its breadth was 25 in., depth to steel 30.5 in., span 23.5 ft., and percentage of steel 1.25. Only a portion of each curve is shown; the dot on each corresponds to one-fourth the ultimate applied load. The dashed lines are the graphs of the deflection formulas (Arts. 159 and 160) for the corresponding beams; 8 is the value of n used in groups 1 and 2.

162. *Effect of Stirrups and Bent-up Rods.*—Stirrups and bent-up rods do not affect the stiffness of the beam materially for working loads; but they do increase the ultimate deflection as well as strength. Bach's tests clearly show this to be true, for example:

(1) Column a of the adjoining table (No. 21) gives the average deflections for three beams (numbers 7, 13, and 14) corresponding to the loads tabulated; the beams were reinforced with a single straight rod (p about .9%). Column b gives the average deflections for another set of three (29, 32, and 37); these were reinforced like the first set but with sixteen stirrups added. The fourth column gives the percentage differences between the deflections of the two sets of beams up to 4000 kg. The average ultimate deflections of the two sets were 1.78 and 2.3 mm., and the ultimate loads 18,900 and 23,250 kg. respectively.

(2) Column A of the same table gives the average deflections for a set of beams (40, 43, and 45) which were reinforced with three straight rods (p=.55%); and B the average deflections for a set (49, 51, and 53) reinforced like the first, but two of the rods were bent up at each end. The last column gives the percentage differences between the average deflections of the two sets of beams. The average ultimate deflection of sets A

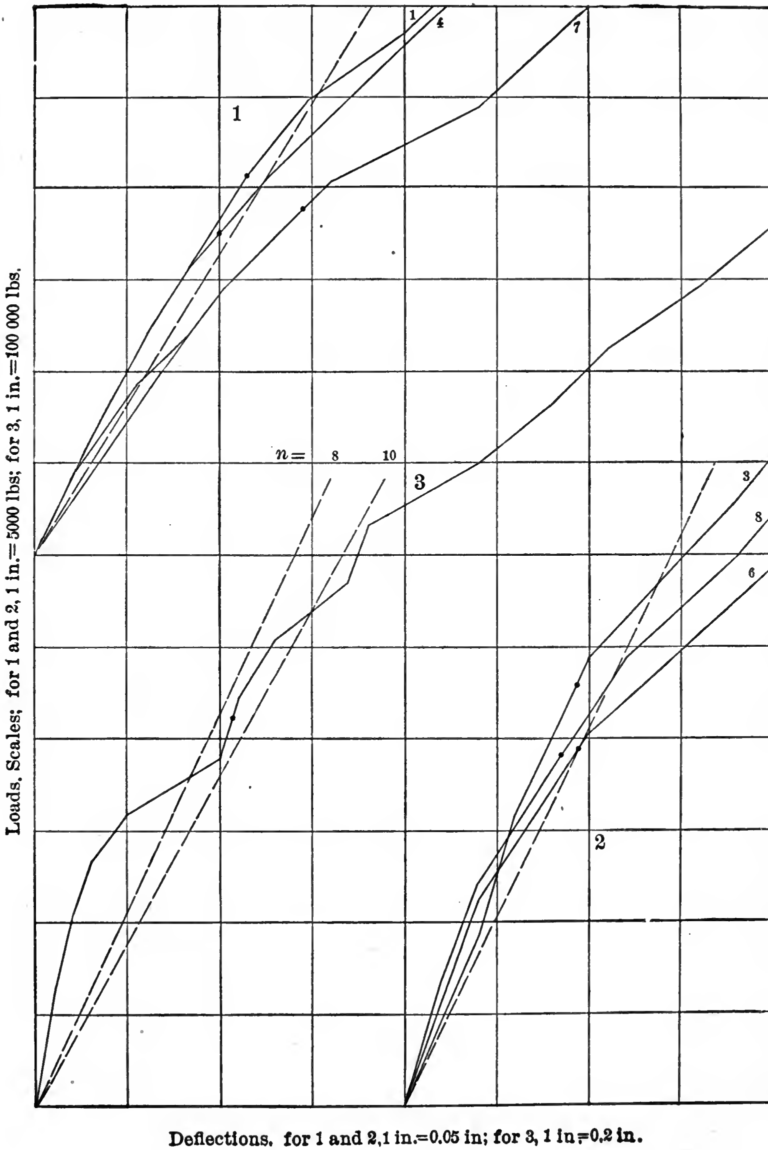

1
4
7
1
n= 8 10
3
3
8
6
2
Loads. Scales; for 1 and 2, 1 in.=5000 lbs; for 3, 1 in.=100 000 lbs.
Deflections. for 1 and 2, 1 in.=0.05 in; for 3, 1 in=0.2 in.

FIG. 6.

TABLE NO. 21.

DEFLECTIONS OF RECTANGULAR BEAMS.

Load, (Kilos).	Deflection (millimeters).					
	a.	*b.*	Diff.	*A.*	*B.*	Diff.
500	.052	.052	0%	.048	.050	+4.0%
1000	.110	.107	−2.7	.107	.110	+2.8
1500	.175	.165	−5.7	.167	.173	+3.6
2000	.245	.232	−4.9	.232	.248	+6.8
2500	.322	.307	−4.6	.308	.330	+6.9
3000	.428	.417	−2.6	.403	.403	+6.7
3500	.608	.580	−4.6	.538	.585	+8.7
4000	.793	.767	−3.3	.775	.902	+14.0

and *B* were 3.38 and 3.45 mm. and their average ultimate loads 8250 and 8600 kg., respectively.

(3) The numbered columns in the adjoining table (No. 22) give the average deflections of six sets of T-beams, three in

TABLE NO. 22.

DEFLECTIONS OF T-BEAMS.

Loads (kilos).	Deflections (millimeters).						
	1	2	3	4	5	6	Diff.
2000	.090	.087	.088	.093	.092	.095	9.2%
4000	.193	.183	.180	.190	.190	.192	7.2
6000	.307	.290	.287	.303	.293	.303	10.5
8000	.435	.400	.398	.422	.407	.420	6.0
10000	.577	.535	.568	.553	.553	.563	1.8

each set, for the loads tabulated. The beams were alike except as to reinforcement. Beams of set 1 were reinforced with three straight rods ($p=.8\%$); set 2 like 1 and 24 stirrups; set 3 like 1 and 48 stirrups; set 4 with five rods ($p=.87\%$), four bent up at each end; set 5 like 4 and 24 stirrups; and set 6 like 5 except that a hook was formed at each end of the fifth rod. The horizontal lines in the table are drawn to

correspond to one-quarter ultimate loads. The last column of the table gives the greatest percentage difference for the various working loads. The average ultimate deflections were 2.4, 3.2, 3.8, 6.0, 5.8, and 9.4 mm.; the average ultimate loads 23,000, 30,500, 37,800, 33,300, 41,000, 46,000 kg., respectively. All average ultimate deflections are not reliable.

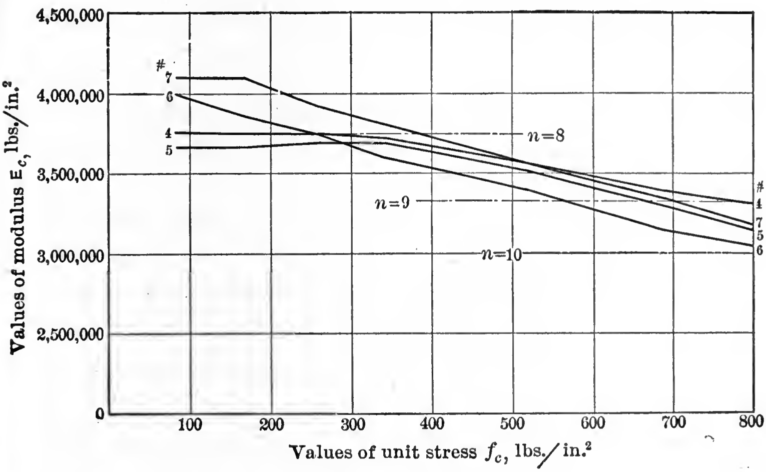


FIG. 7.

163. The Value of n for Deflection Formulas.—As explained in Arts. 158 and 159, the value of the modulus of elasticity to be used in deflection formulas should correspond not to the greatest unit stress in the concrete but to a fair average of the unit stress at all points in the beam. Fig. 7 shows how the secant modulus varied in four compression specimens representing the concrete of the Bach beams referred to in Art. 161. It was a 1 : 4 gravel concrete and the specimens were about eight months old when tested. The curves show that for unit stresses as high as 600 lbs/in² the moduli averaged over 3½ million ($n=9$), and for the fair average unit-stress in Bach's beams under working loads n would be about 8.

CHAPTER VII.

COLUMNS.

THEORY OF STRENGTH OF COLUMNS.

164. **The Relative Length of Concrete Columns.**—Concrete columns need rarely be calculated as long columns. In ordinary construction the ratio of length to least width will seldom exceed 12 or 15, while the results of tests indicate little or no difference in strength for ratios up to 20 or 25. It will be desirable, then, to determine first the strength of a reinforced column considered as a short column. If the conditions require it a general column formula may then be applied to provide for cases where the length is excessive. The Joint Committee recommends a maximum value of 15 for the ratio of length to least width for reinforced columns.

165. **Methods of Reinforcement.**—There are three general methods of reinforcing columns namely:

(1) By longitudinal reinforcement consisting of rods or shapes;

(2) By hoops or spirally wound metal closely spaced; and

(3) By both longitudinal and hoop reinforcement.

Longitudinal reinforcement aids the concrete by carrying a part of the load directly, the stresses in the two materials being proportional to their moduli of elasticity. Hoops and bands support the concrete laterally, preventing lateral expansion to a greater or less degree, and thus strengthening the concrete. Usually both systems are more or less combined, longitudinal rods or structural shapes being fastened together at intervals by circumferential bands of some sort, and on the other hand hoops or spiral wire being conveniently held in

place by longitudinal rods. Experiments show that both types of reinforcement are effective in raising the ultimate strength of a column, but a consensus of opinion has not yet been reached as to the proper allowance to be made in working formulas.

166. Columns with Longitudinal Reinforcement.—So long as the steel and concrete adhere the relative intensities of stress in the two materials will be proportional to their moduli of elasticity, using the secant modulus as explained in Art. 24.

Let A = total cross-section of column;
A_c = cross-section of concrete;
A_s = cross-section of steel;
p = ratio of steel area to total area $= A_s/A$;
f_c = unit stress in concrete;
f_s = unit stress in steel;
f = average unit stress in entire section;
n = ratio of moduli of steel and concrete at the given stress f_c, $= E_s/E_c$;
P' = total strength of a reinforced column for the stress f_c.

Then $$P' = f_cA_c + f_sA_s = f_c(A - pA) + f_cnpA,$$

whence $$P' = f_cA[1 + (n-1)p], \quad (1)$$

from which $$f = P'/A = f_c[1 + (n-1)p]. \quad (2)$$

The relative increase in strength caused by the reinforcement is then

$$\frac{f - f_c}{f_c} = (n-1)p. \quad (3)$$

The elastic limit of the steel, if low, may affect the ultimate strength of the column. The value of P' is then not greater than

$$P' = f_cA_c + f_{el}A_s, \quad (4)$$

and $$f = f_c(1-p) + f_{el}p, \quad (5)$$

in which f_{el} is the elastic limit strength of the steel. (See Art. 172 for further discussion of this question.)

Eq. (2) is convenient to use in determining the relative strength of a reinforced as compared to a plain concrete col-

umn for a given percentage of steel. Thus, if $p=1\%$ and $n=15$, we have $f/f_c=1+0.14=1.14$. Thus a reinforcement of 1% increases the strength by 14%.

From these relations it is seen that the relative increase in strength caused by a given amount of reinforcement depends on the value of n and is greater the larger n is.

The economy of steel reinforcement is also dependent upon the working stresses permissible in the concrete since $f_s=nf_c$. The following table shows the various working stresses in the steel corresponding to various values of working stress in the concrete and to various values of the modulus E_c; there is given also the percentage increase in strength for each one per cent of steel.

TABLE No. 23.

LONGITUDINAL REINFORCEMENT OF COLUMNS.

f_c, lbs./in.²	E_c, lbs/in²	Ratio of Moduli, n	f_s, lbs/in²	Percentage Increase in Strength for each 1% Reinforcement.
300	750,000	40	12,000	39
	1,000,000	30	9,000	29
	1,500,000	20	6,000	19
	2,000,000	15	4,500	14
400	1,000,000	30	12,000	29
	1,500,000	20	8,000	19
	2,000,000	15	6,000	14
	2,500,000	12	4,800	11
500	1,000,000	30	12,000	29
	1,500,000	20	9,000	19
	2,000,000	15	7,500	14
	2,500,000	12	6,000	11
600	1,500,000	20	12,000	19
	2,000,000	15	9,000	14
	2,500,000	12	7,200	11
	3,000,000	10	6,000	9
800	2,000,000	15	12,000	14
	2,500,000	12	9,600	11
	3,000,000	10	8,000	9
	3,500,000	8.6	6,900	7.6

From this table the relation among the various quantities may be clearly understood. It is to be noted that the working stresses in the steel must be relatively low except in the unusual combination of high working stresses in the concrete with low modulus. High-grade concrete, permitting high working stresses, will have a high modulus.

The foregoing theoretical relations are not fully borne out by tests, as shown in Art. 172. The presence of the longitudinal reinforcement, especially if in the form of riveted structural shapes, tends to interfere with the production of as dense a concrete as in the case of a plain concrete column, and so lowers the strength of the concrete to some extent. For further discussion see Art. 172.

167. Columns with Hoop Reinforcement.—Whenever a material which is subjected to compression in one direction is restrained laterally, then lateral compressive stresses are developed which tend to neutralize the effect of the principal compressive stresses and thus to increase the resistance to rupture. Were the compressive stresses equal in all directions there would be no rupture (as there would be no shear). The strengthening effect of lateral banding depends then upon the rigidity of the bands, that is, upon the amount of steel used and its closeness of spacing. Its elastic limit may also affect the ultimate strength of the column.

On the basis of the relative lateral and longitudinal deformation of the concrete (Poisson's ratio) it is possible to deduce a theoretical relation between the lateral and the longitudinal stresses, and thence the portion of the longitudinal stress remaining unbalanced. Let μ = Poisson's ratio, f_c = unbalanced or excess of longitudinal over lateral compressive unit stress, f_c' = total longitudinal unit stress, f_s = unit tensile stress in steel, p = steel ratio = ratio of volume of steel to volume of concrete.

We find approximately

$$f_c' = f_c\left(1 + \frac{\mu n p}{2}\right), \quad \ldots \ldots \quad (1)$$

and $$f_s=\mu n f_c.^* \quad \text{(2)}$$

* *Demonstration.* Let μ = Poisson's ratio; p = steel ratio considered as a thin cylinder of equivalent area surrounding the concrete; A_s = cross-section of this steel cylinder; r = radius. Then

$$A_s=p\pi r^2 \quad \text{and} \quad \text{thickness of cylinder}=\frac{p\pi r^2}{2\pi r}=p\frac{r}{2}.$$

With no steel banding the stress f_c' would cause a proportionate lateral swelling of $\frac{f_c'}{E_c}\mu$. If the actual stress in the steel is f_s then the compression per sq. in. developed in the concrete by the steel reinforcement $=f_s p\frac{r}{2}\div r=\frac{f_s p}{2}$. This compression caused by the banding is equal in all horizontal directions, and has the same effect on distortion as two pairs of equal compressive forces acting on two sets of faces of a cube. The resultant lateral compression due to these horizontal forces is equal to $\frac{f_s p}{2E_c}(1-\mu)$. Combining this compression with the lateral swelling caused by f_c' we have the net lateral deformation equal to $\frac{f_c'}{E_c}\mu-\frac{f_s p}{2E_c}(1-\mu)$. This net deformation must equal the actual deformation in the steel under the stress f_s, which is $\frac{f_s}{E_s}$ or $\frac{f_s}{nE_c}$. Hence we have

$$\frac{f_c'}{E_c}\mu-\frac{f_s p}{2E_c}(1-\mu)=\frac{f_s}{nE_c}.$$

A part of f_c' may be considered to be balanced by the lateral compression of $\frac{f_s p}{2}$; it is the unbalanced portion only which is significant. Call this unbalanced portion f_c; then $f_c'=f_c+\frac{f_s p}{2}$. Then eliminating f_s from these two equations we find for f_c' the value

$$f_c'=f_c\left(1+\frac{np\mu}{np(1-2\mu)+2}\right) \quad \text{(a)}$$

We also have

$$f_s=\frac{2\mu n}{np(1-2\mu)+2}f_c. \quad \text{(b)}$$

For ordinary values of p eqs. (a) and (b) are reduced approximately to

$$f_c'=f_c\left(1+\frac{\mu np}{2}\right) \quad \text{(1)}$$

and

$$f_s=\mu n f_c. \quad \text{(2)}$$

Taking Poisson's ratio at $\frac{1}{8}$, eqs. (1) and (2) become $f_c' = f_c(1+np/16)$, and $f_s = \frac{1}{8}nf_c$. Comparing these equations with those of Art. 166 it appears that within the limit of elasticity the hoop reinforcement is much less effective than longitudinal reinforcement; in fact it would seem that very little stress can be developed in the steel under elastic conditions as here assumed. Such reinforcement is, however, quite effective in increasing the *ultimate* strength of a column.

Results of tests appear to accord in a general way with these theoretical relations. Hooped columns show about the same behavior as plain concrete columns up to a load nearly equal to the ultimate strength of the plain concrete. Under further loading the concrete is prevented by the banding from actual failure, but continues to compress and to expand laterally, increasing the tension in the bands, the elasticity of the bands rendering the column in large degree still elastic. Final failure occurs upon the breakage of the bands or their excessive stretching. Banded columns thus exhibit a toughness or ductility much greater than other forms, but without a corresponding increase in stiffness under low loads. Ultimate failure is likely to be long postponed after the first signs of rupture, and the column will sustain greatly increased loads even after the entire failure of the shell of concrete outside the bands.

168. Columns with Both Longitudinal and Hoop Reinforcement.—From the theoretical considerations of the preceding article it would appear that the addition of bands or hoops to columns having longitudinal reinforcement would not have much effect upon the deformation of such columns until the ordinary elastic limit strength of the concrete has been passed. The effect of such hooping would be to maintain the integrity of the concrete beyond the usual limit of deformation and so enable the longitudinal steel to be stressed to a higher value. Results of tests discussed in Art. 174 bear out these conclusions.

169. Concrete Columns as Long Columns.—Concrete columns should preferably not be used with a slenderness ratio exceed-

ing about 15. Up to this value the tests indicate little or no effect due to lateral flexure. Variation in strength with length is due more to the increased likelihood of the occurrence of weak spots in such a non-homogeneous material as concrete than to increased flexibility. If slender members are desired for unimportant parts, some form of column formula must be used. Until more data are available from tests, the authors would propose the use of the theoretical form of Rankin's formula which is, for pivoted ends,

$$P''=\frac{P'}{1+\frac{f}{\pi^2 E}\left(\frac{l}{r}\right)^2}, \quad \ldots \ldots \quad (1)$$

in which P' is the strength of a short column, and f and E are the ultimate strength and the modulus of elasticity of the concrete. This formula gives results materially too low when applied to steel columns, but it is believed that it is not too conservative for material like concrete. The value of f/E should, for conservative design, be taken at its maximum rather than minimum value, say $^1/_{1000}$, giving finally the formula

$$P''=\frac{P'}{1+\frac{1}{10,000}\left(\frac{l}{r}\right)^2}. \quad \ldots \ldots \quad (2)$$

For fixed ends the constant in the denominator may be made $^1/_{20000}$ giving

$$P''=\frac{P'}{1+\frac{1}{20,000}\left(\frac{l}{r}\right)^2}. \quad \ldots \ldots \quad (3)$$

It may be observed that if formulas be derived from the Rankine-Gordon formulas for steel columns, by taking account of the difference in ultimate strength and modulus of elasticity of the materials, the resulting formulas would contain constants of very nearly the same value as for steel, namely,

$^1/_{18000}$ and $^1/_{36000}$. The low values above given represent a larger degree of safety, which is to be desired. For a value of l/r of 100, or a length of about 30 diameters, the formula for pivoted ends gives an ultimate strength of two-thirds that of the short column. Because of the fact that it is difficult to secure thoroughly homogeneous concrete, and that variations in quality will affect the strength of long columns more seriously than any other structural form long columns should generally be avoided.

TESTS OF COLUMNS.

170. Tests of Plain Concrete Columns.—Results of tests on plain concrete columns of lengths 10 to 15 diameters show considerably less strength per square inch than the same concrete in the short cylindrical specimen. This is due not so much to the effect of flexibility, as such a column is very rigid, but rather to the increased effect of the non-homogeneous character of the material and to various unavoidable imperfections and irregularities. Plain concrete columns of dense material fail in the same general manner as the short specimen, that is, by a sudden shear failure, leaving the broken ends in a conical or pyramidal form at approximately 45 degrees inclination. Columns made of poor concrete are likely to fail by gradual crushing.

Tests of columns 8 feet long by 10 to 12 inches in diameter made at the Watertown Arsenal gave an average strength for 1 : 2 : 4 concrete, at six months, of 1600 to 1700 lbs/in^2.*

Table No. 24 gives results of tests made at the University of Illinois.

In general, it was found that the richer mixtures tended to fail by true shear failures, while the poorer mixtures generally failed by gradual crushing. The very superior results obtained on the 1 : $1\frac{1}{2}$: 3 mixture as compared with the 1 : 2 : 4 mixture, or poorer, should be noted. It shows the value of

* Tests of Metals, 1904.

TABLE NO. 24.

TESTS OF PLAIN CONCRETE COLUMNS.

UNIVERSITY OF ILLINOIS, 1907.*

All columns were 12 in. in diameter by 10 ft. long.

Group.	Col. No.	Kind of Concrete.	Crushing Strength, Lbs/in². Individual Tests.	Crushing Strength, Lbs/in². Average for Group.	Age of Specimen, Days.
1	111 112	1:1½:3	2120 2480	2300	66 62
2	101 102 103 104 105 108 109	1:2:4	1165 2000 2210 1590 1945 1460 1810	1740	58 69 65 64 62 72 64
3	116 117	1:3:6	955 1110	1030	61 62
4	121 122	1:4:8	575 575	575	63 63
5	110 128 129 163 164 168	1:2:4	1925 1845 1770 2680 2160 1770	2025	203 194 181 187 187 201
6	21 22	1:2:3¾	2650 2770	2710	12 mo. 16 mo.

the use of rich mixtures for columns, the increase in strength over the 1 : 2 : 4 concrete being about 32% while the increase in cost would not be over 10 or 15%. The great variation in individual tests in Table No. 24 should be noted, the results for group 2 varying from 33% below to 27% above the average. Results of comparative tests on short cylinders of 1 : 2 : 4 concrete, stored in damp sand for 9 to 11 months, gave an average crushing strength of 2650 lbs/in²; tests on 12-inch cubes

* From Bulletin No. 20, Eng. Exp. Sta., University of Illinois.

stored in air at age of sixty days gave an average value of about 1950 lbs/in², and at age of about two hundred days, of 2350 lbs/in².

Tests made at the University of Wisconsin in 1908 indicate that with careful workmanship and testing an average value of about 2000 lbs/in² can be obtained in sixty days on 1 : 2 : 4 concrete, the results there obtained being very uniform. Short cylindrical specimens of the same concrete gave a strength of 2360 lbs/in².

From these results we may conclude that for lengths up to 10 diameters, the strength of well-made columns of 1 : 2 : 4 concrete centrally loaded is at least 85% of the strength of short cylindrical specimens.

It can hardly be considered good practice to use unreinforced concrete for lengths as great as represented in the foregoing tests. The Joint Committee make no recommendations for unit stress for columns of plain concrete exceeding 4 diameters in length.

171. Tests of Columns with Longitudinal Reinforcement.—*General Behavior of Columns.*—Columns with longitudinal reinforcement only exhibit about the same characteristics in a test as columns of plain concrete. When the stress on the concrete has reached its ordinary ultimate strength, the concrete fails on shearing planes, the rods bending or buckling at the same time. Fig. 1 illustrates a typical failure.

It is important to note that in this type of column the steel gives very little aid in preventing the concrete shearing out; it merely serves to carry part of the load until the concrete is overloaded. With ordinary materials the deformation reached at the ultimate strength of the concrete brings the stress in the steel up to about its elastic limit and sometimes beyond. For example, if the ultimate strength of the concrete is 3000 lbs/in², and E_c at rupture$=2,000,000$, the stress in the steel corresponding to the ultimate strength of the concrete is $3000\times15=45,000$ lbs/in², which may exceed its elastic limit. Inasmuch as the steel rapidly deforms as soon as the elastic

limit is reached, this limit may be taken as the maximum possible stress in the steel. In the above case, for example, if the elastic limit of the steel be 40,000 lbs/in², the ultimate strength of the column may be taken at $f_cA_c+f_sA_s$, where $f_c=3000$ and $f_s=40,000$ lbs/in².

FIG. 1.—Failure of Column with Longitudinal Reinforcement.

FIG. 2.—Failure of Column with Hoop Reinforcement.

In Fig. 3 curves C and D are typical stress-strain diagrams of a plain concrete column and of a column reinforced with longitudinal rods. The ordinates represent the average stress per square inch on the entire column. The effect of the steel in D is to carry part of the load, giving less strain for the same total load. The ultimate deformations are not greatly different, that of the reinforced column generally being a little the

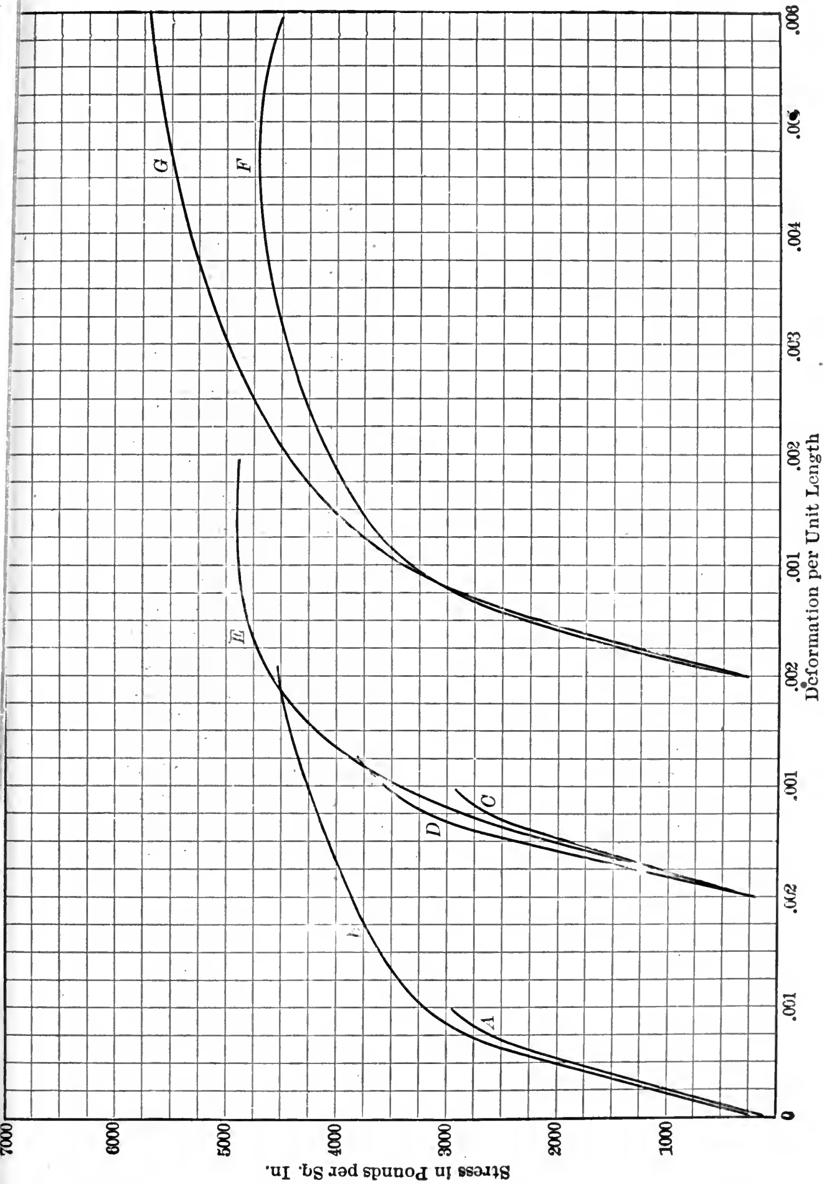


FIG. 3.—Typical Stress-strain Diagrams of Reinforced Columns.

greater. In both cases the failure is sudden and the total deformation small. The curves are similar in form to the ordinary compression curve for concrete.

172. *Results of Tests.*—The following is a summary of a series of tests on columns, reinforced with longitudinal rods only, made at the Massachusetts Institute of Technology.* The concrete was 1 : 3 : 6 proportions, the rods from $\frac{3}{4}$ to $1\frac{1}{4}$ inches in diameter, partly plain rods and partly twisted. The columns ranged from 6 to 17 feet in length, and were about thirty days old.

For 8×8 inch columns, average length 12.4 feet:

Reinforcement, per cent =	1 56	2.44	3.51	6.25
Average strength, lbs/in² =	1904	2267	2535	3140

For 10×10 inch columns, average length 11.0 feet:

Reinforcement, per cent =	1.00	1.56	2.25
Average strength, lbs/in² =	2145	2452	2870

Taking $n = 20$, at ultimate strength, these values would be fairly well represented by the theoretical equations (eq. (2), Art. 166), $f = 1470\ (1 + 19p)$ for the small columns and $f = 1300(1 + 19p)$ for the large columns. The effect of size of column is very apparent.

Table No. 25 contains results of tests made by Professor A. N. Talbot at the University of Illinois. The columns were made of 1 : 2 : $3\frac{3}{4}$ concrete and plain steel of 39,800 pounds per square inch elastic limit. The age was from 59 to 71 days. Comparing the reinforced with the plain concrete, the average strength of the 12×12 inch columns with 1.2 per cent reinforcement is about 1.17 times as great, and the 9×9 inch columns with 1.5 per cent reinforcement is about 1.10 times as great. These tests indicate a less effect of reinforcement than some of the other tests quoted. The smaller cross-section of the columns containing the larger amount of reinforcement

* Trans. Am. Soc. C. E.. Vol. L, 1903, p. 487.

Table No. 25.

TESTS OF COLUMNS WITH LONGITUDINAL REINFORCEMENT.

University of Illinois, 1906.*

No.	Length.	Cross-section.	Reinforcement. Kind.	Reinforcement. Per cent.	Crushing Strength. Pounds per sq. in. Individual Test.	Crushing Strength. Pounds per sq. in. Average of Group.
1	12 ft.	12″×12″	4 $\frac{3}{4}$-in. rods	1.20	1587	1809
3			4 $\frac{3}{4}$-in. rods 12 $\frac{1}{4}$ in. ties	1.21	1862	
7			4 $\frac{3}{4}$ in. rods	1.21	1850	
11			4 $\frac{3}{4}$-in. rods 12 $\frac{1}{4}$-in. ties	1.21	1936	
2	12 ft.	9″×9″	4 $\frac{5}{8}$-in. rods	1.52	1577	1710
6	"		4 $\frac{5}{8}$-in. rods	1.52	1600	
10	"		4 $\frac{5}{8}$ in. rods 12 $\frac{1}{4}$-in. ties	1.50	1280	
12	9 ft.		4 $\frac{5}{8}$-in. rods 9 $\frac{1}{4}$-in. ties	1.48	2335	
14	12 ft.		4 $\frac{5}{8}$-in. rods 12 $\frac{1}{4}$-in. ties	1.50	1367	
16	9 ft.		4 $\frac{5}{8}$-in. rods 9 $\frac{1}{4}$-in. ties	1.49	1607	
17	6 ft.		4 $\frac{5}{8}$-in. rods	1.47	2206	
5	12 ft.	12″×12″	Plain	0	1710	1550
8	"	9″×9″			2004	
9	"	12″×12″			1610	
13	"	" "			1709	
15	6 ft.	" "			1189	
18	"	9″×9″			1079	

may have been the cause of the lower strength of this group. It is important to note the wide variation in the individual results of these and other tests; they indicate what may be expected in practice, and show clearly the necessity of adopting conservative values of working stress. Careful measurement of distortions showed that the ratio of stress in steel to stress in concrete varied from about 14 at the beginning to about 27 at rupture, taking average values. The low values for ultimate strength of the reinforced columns appeared to be due to a lower actual crushing strength of the concrete in these columns than in the plain columns.

* Bulletin No. 10, Engineering Exp. Sta., 1907.

From Table No. 29, p. 237, we have the following average values for the strength of columns of 1 : $1\frac{1}{2}$: 3 concrete with longitudinal reinforcement only.

Per cent Reinforcement.	Strength; lbs/in².
0	2745
1.00	3750
2.03	3690
4.07	4535

These columns were 9 feet 4 inches long between enlarged heads, by about 20 inches diameter. The strength of cylindrical test specimens was 2940 lbs/in². The above values are fairly represented by the formula $f=3000(1-p)+40{,}000p$, the stress of 40,000 lbs/in² being slightly above the elastic limit strength of the steel. (See eq. (5), Art. 166.) The same result is given by the theoretical formula $f=f_c[1+(n-p)]$ (eq. (2)), with $f_c=3000$ and $n=13\frac{1}{3}$. As this value of n would be about right for these columns it appears that the ultimate strength of the concrete and the yield point of the steel must have been reached at about the same deformation.

From Table 28, p. 234, we have the following values for 1 : 2 : 4 concrete:

Compression cylinders	2360 lbs/in²
Plain concrete columns.	2070 lbs/in²
Columns with 2.35% reinforcement . .	2470 lbs/in²

The strength of the plain concrete column is thus 87% of that of the short compression specimen and of the reinforced column 20% more than the plain column. By eq. (3) and with $n=15$, the latter would be $14\times 2.35=33\%$ more instead of 20%. For $n=10$ the formula gives 21.2% increase.

173. Tests of Hooped Columns.—*General Behavior of Columns.*—As shown in Art. 167, when a compression member is reinforced by bands or hoops closely spaced, such reinforcement will raise the ultimate strength by preventing lateral expansion under the compressive forces. It was also shown

that under this system of reinforcement the steel cannot be stressed to any considerable extent under loads below the usual elastic limit strength of the concrete. This limit being exceeded, however, the banding becomes very effective in holding the concrete together so that it will endure large deformations without rupture, thus increasing greatly its ultimate strength. Longitudinal reinforcement is also used with hoops or bands. Such reinforcement will receive stress in proportion to the longitudinal deformations and will thus be more effective at low loads than the bands. Results of tests on both forms of columns are here given. The general behavior of hooped columns is well illustrated in Fig. 2 and in the curves of Fig. 3. These are typical stress-strain diagrams representing some of the tests of Table No. 28.

Curves *A* and *B* show a comparison between a plain concrete column and a hooped column. The deformation is practically the same up to the crushing point of the plain concrete. Beyond this the hooped column undergoes a greatly increased deformation before rupture takes place. Curves *C*, *D*, and *E* represent a plain concrete column, a column with 2% longitudinal reinforcement only, and a column with 2% longitudinal and 1% spiral or hoop reinforcement. Note the effect of the longitudinal rods in stiffening and strengthening the column, and that the addition of the spiral reinforcement has the effect of increasing very greatly the possible deformation and the ultimate strength but not the behavior of the column within the elastic strength. Thus we see in both cases that the general effect of hoop reinforcement is greatly to increase the ultimate strength and toughness of a column, but without changing the behavior materially within the elastic range, or for loads within the range of the ordinary ultimate strength of the unhooped columns.

174. *Results of Tests.*—In 1902 and 1903 Considère * published certain tests made on columns reinforced by spirally

* Génie Civil, 1902.

wound wire and by longitudinal rods or wire. His most important results were those obtained upon a number of octagonal columns 5.9 inches short diameter. As a result of these and other tests, as well as from a theoretical basis, he came to the conclusion that steel in the form of spiral reinforcement was 2.4 times as efficient in increasing the ultimate strength of a column as steel in the form of longitudinal reinforcement, presuming the spacing of the wire to be not great ($\frac{1}{4}$ to $\frac{1}{10}$ of the diameter of the spiral) and that ordinary mild steel be used. It was found also desirable to use a small amount of steel in the form of longitudinal reinforcement. Tests on the elastic properties showed considerable deformation and set, but after the first application of load the column was relatively rigid, with greatly increased value of E.

Table No. 26 gives the results of a series of tests on hooped columns made by Professor Talbot, in 1907. Two forms of hooping were used, electrically welded bands 1 inch wide and of various gage thickness, and spirally wound wire at a pitch of 1 inch. The steel used in the bands had a yield point of about 48,000 lbs/in². The wire was of two kinds, high carbon and mild steel. The former had a yield point of 115,000 lbs/in² for the $\frac{1}{4}$-inch size and 60,000 lbs/in² for the No. 7; the latter had yield points for the same sizes of 54,000 and 38,500 lbs/in², respectively. The columns were 10 feet long by 12 inches in diameter. A thin film of mortar covered the hooping.

As to ultimate strength, the results may be compared by groups with those for plain concrete given in Table No. 24, group 2. The figures are brought together as shown in the table on page 232.

Taking the elastic-limit strength as a basis, the theoretical effect of 1% of longitudinal steel would be, in the case of the band steel, 480 lbs/in², the high carbon wire 600 and 1150 lbs/in² and the mild steel about 500 lbs/in². These values are not far from those given in the last column of the table, thus showing that the effect of the spiral steel was in this case about the same

TABLE NO. 26.

TESTS OF HOOPED COLUMNS.

UNIVERSITY OF ILLINOIS, 1907.*

Concrete 1:2:4; age, from 56 to 69 days; length, 10 ft.; diam., 12 ins.

Group.	Col. No.	Reinforcement. Kind.	Reinforcement. Size and Spacing.	Reinforcement. Per Cent.	Crushing Strength, Lbs/in.² Individual Tests.	Crushing Strength, Lbs/in.² Average of Group.
1	131	Electrically welded bands.	No. 16, 2 in. c.–c.	1.08	2384	2239
	132			1.08	2150	
	133			1.05	2182	
2	136		No. 12, 2 in. c.–c.	2.08	2860	2877
	137			2.07	2660	
	138			2.12	3110	
3	146		No. 8, 2 in. c.–c.	3.22	3000	3202
	147			3.20	3715	
	148			3.20	2890	
4	143		No. 12, 3 in. c.–c.	1.39	2735	2735
5	141		No. 12, 4 in. c.–c.	1.02	2275	2226
	142			1.02	2178	
6	171	High carbon wire spiral.	No. 7	0.85	2503	2505
	172			0.85	2506	
7	181		¼ in.	1.73	2718	3437
	182			1.67	3800	
	183			1.68	3793	
8	176	Mild steel wire spiral	No. 7	0.84	2080	2168
	177			0.85	2203	
	178			0.84	2220	
9	186		¼ in.	1.64	2068	2736
	187			1.71	3404	
	188			1.61		

* Bull. No. 20, Eng. Exp. Sta., Univ of Ill.

COMPARISON OF HOOPED AND PLAIN CONCRETE COLUMNS.

UNIVERSITY OF ILL. TESTS.

Group.	Kind of Reinforcement.	Average Amount of Reinforcement, Per Cent.	Average Ultimate Strength, Lbs/in².	Excess over Plain Concrete, Lbs/in².	
				Total.	Per 1% Reinforcement.
2	Plain concrete	0	1740	....	....
1	Bands	1.07	2239	599	560
2	Bands	2.09	2877	1137	540
3	Bands	3.21	3202	1462	450
4	Bands	1.39	2735	995	710
5	Bands	1.02	2226	486	480
6	High carbon wire	0.83	2505	765	920
7	High carbon wire	1.69	3437	1697	1000
8	Mild steel wire	0.84	2168	428	510
9	Mild steel wire	1.65	2736	996	600

as would be obtained by longitudinal steel if stressed to its elastic limit.

Tests made at the Watertown Arsenal in 1905 * showed results very similar to those quoted above. The reinforcement consisted of riveted bands 1.5×0.12 inches and longitudinal angle bars $1 \times 1 \times \frac{1}{8}$ inch. The columns were 1 : 2 : 4 concrete 5 and 6 months old and were $10\frac{1}{2}$ inches in diameter by 8 feet long. The entire column was inclosed by the bands. The results are given in Table No. 27.

The additional strength of the hooped columns over the plain concrete for each 1% of reinforcement was 819 lbs/in² for 13 hoops, 1120 lbs/in² for 25 hoops and 1140 lbs/in² for 47 hoops. The additional strength due to the angle bars over the same column without the bars was 797 lbs/in² for the one with 13 hoops and 761 lbs/in² for that with 25 hoops. These values are about equal to the highest obtained by Talbot.

* Tests of Metals, 1916.

TABLE No. 27.

TESTS OF HOOPED COLUMNS.

WATERTOWN ARSENAL, 1905.

Reinforcement. Kind.	Per Cent Hoops.	Per Cent Longitudinal.	Strength, Lbs/in².
Plain concrete	....	....	1413
13 hoops	1.0	.0	2232
13 hoops, 4 L's	1.0	1.0	3029
25 hoops	1.8	.0	3428
25 hoops, 4 L's	1.8	1.0	4189
47 hoops	3.4	.0	5289

The results of an extensive series of tests of reinforced columns made by Mr. M. O. Withey at the University of Wisconsin are given in Table No. 28 and in Figs. 4 and 5.* Columns W were unreinforced; E were reinforced by longitudinal bars only with wire ties widely spaced; B were reinforced by latticed angles forming a square 8×8 inches in section; the remaining columns by spiral wire and varying proportions of longitudinal reinforcement wired to the spirals on the inner side. The diameter of the columns was made equal to the outside diameter of the spirals, but the calculations have been made with reference to the inside diameter of 10 inches. The yield point of the steel reinforcement was 38,000 to 45,000 lbs/in² for the longitudinal steel and 80,000 to 105,000 lbs/in² for the spiral steel.

The characteristics of the various columns are well brought out in Fig. 4. The "toughening" effect of the hooping is shown by the curves H_1 and I_1. Also, comparing H_1 and I_1, note that both contain the same percentage of hoop reinforcement, but I_1 has 3.78% vertical reinforcement and H_1 none. A larger percentage of hoop reinforcement would undoubtedly have extended both of these curves further towards the right and also somewhat higher, but would not have affected the

* Bul. No. 466, Univ. of Wis., 1911.

TABLE No. 28.

TESTS OF REINFORCED COLUMNS.

UNIVERSITY OF WISCONSIN.

Length of columns $A \ldots C = 120''$, $H \ldots U = 100''$. Age 2 months.

No.	Reinforcement.			Concrete.		Cross-section, sq.in.	Average Strength of Columns, lbs/in².		
	Kind.	Per Cent Vertical.	Per Cent Lateral.	Mix.	Compressive St'nth of Cyl., lbs/in²		At Yield-point.	At Max. Load.	No. of Tests.
W	None.	0	0	1 : 2 : 4	2420	86.6	. . . .	2600	3
E	9 ⅝″ rods with ¼″ ties, 1 ft. c. to c.	2.35	0.11	1 : 2 : 4	2300	118	. . . .	2470	3
B	4 2″×2″×3/16″ latticed angles.	4.5	. . . .	1 : 2 : 4	2200	64	. . . .	3740	2
C	1″ spiral, 1″ pitch	0	2.0	1 : 2 : 4	2180	78.5	2380	4030	4
D	9 ⅝″ rods and ¼″ spiral, 1″ pitch.	3.50	2.00	1 : 2 : 4	2250	78.5	3580	4750	4
H	No. 7 wire spiral, 2″ pitch	0	0.50	1 : 2 : 3½	1750	78.5	1850	2230	2
G	8 ½″ rods and No. 7 wire spiral, 2″ pitch.	2.0	0.50	1 : 2 : 3½	1760	78.5	2710	3300	2
I	8 11/16″ rods and No. 7 wire spiral, 2″ pitch . .	3.78	0.50	1 : 2 : 3½	2180	78.5	3470	4160	2
J	8 ⅞″ rods and No. 7 wire spiral, 2″ pitch.	6.11	0.50	1 : 2 : 3½	2050	78.5	4240	5120	2
L	No. 7 wire spiral, 1″ pitch	0	1.00	1 : 2 : 3½	1770	78.5	1370	2640	2
K	8 ½″ rods and No. 7 wire spiral, 1″ pitch.	2.0	1.00	1 : 2 : 3½	2000	78.5	2610	3900	2
N	8 11/16″ rods and No. 7 wire spiral, 1″ pitch . .	3.78	1.00	1 : 2 : 3½	1800	78.5	3370	4190	2
M	8 ⅞″ rods and No. 7 wire spiral, 1″ pitch.	6.11	1.00	1 : 2 : 3½	1680	78.5	3760	4680	2
P	8 1″ rods and No. 7 wire spiral, 1″ pitch.	8.0	1.00	1 : 2 : 4	2360	78.5	5666	6920	2
O	8 ⅞″ rods and ¼″ spiral, 1″ pitch.	6.11	1.96	1 : 2 : 4	2480	78.5	4430	6580	2
R	8 1″ rods and ¼″ spiral, 1″ pitch.	8.0	1.96	1 : 2 : 4	2380	78.5	5190	6960	2
Q	8 1⅛″ rods and ¼″ spiral, 1″ pitch.	10.12	1.96	1 : 2 : 4	2300	78.5	5760	7090	2
S	No. 7 wire spiral, 1″ pitch	0	1.00	1 : 1 : 2	4070	78.5	4050	5850	2
T	8 ⅞″ rods and No. 7 wire spiral, 1″ pitch.	6.11	1.00	1 : 1 : 2	4400	78.5	5760	7290	2
V	No. 7 wire spiral, 1″ pitch	0	1.00	1 : 1⅓	4870	78.5	3570	5340	2
U	8 ⅞″ rods and No. 7 wire spiral, 1″ pitch.	6.11	1.00	1 : 1⅓	4550	78.5	5950	8150	2

early portions of the curves. Professor Whitey made a study of the "yield points" of these columns, taking such point to be that point on the curve where the rate of deformation changes rapidly, as at a in curve I_1. It is very nearly that deformation at which plain concrete fails. It was found that the amount of lateral or spiral reinforcement did not materially affect the position of this yield point, but did affect the

ultimate strength. Fig. 5 gives results for both yield point and ultimate strength, plotted so as to enable the effect of

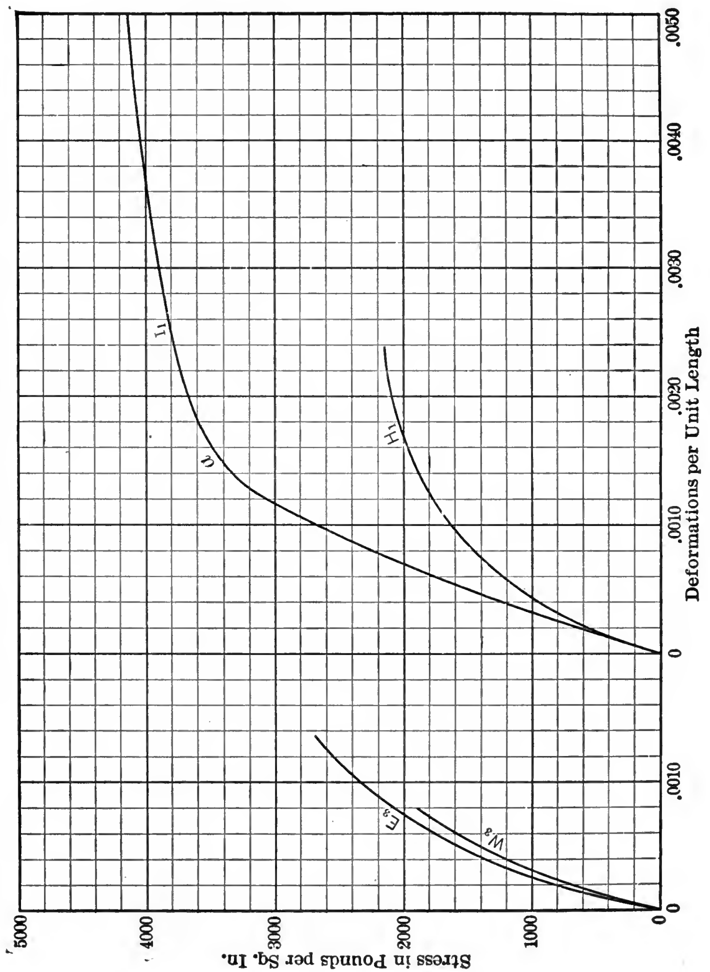


Fig. 4.

the reinforcement to be readily determined. It was found that this "yield point" is closely given by the formula

$$P_1 = A[(1 - p)(f_c + pf_s)],$$

in which f_c is the ultimate strength of the concrete. f_s is the yield point of the longitudinal steel, and p is the steel ratio for

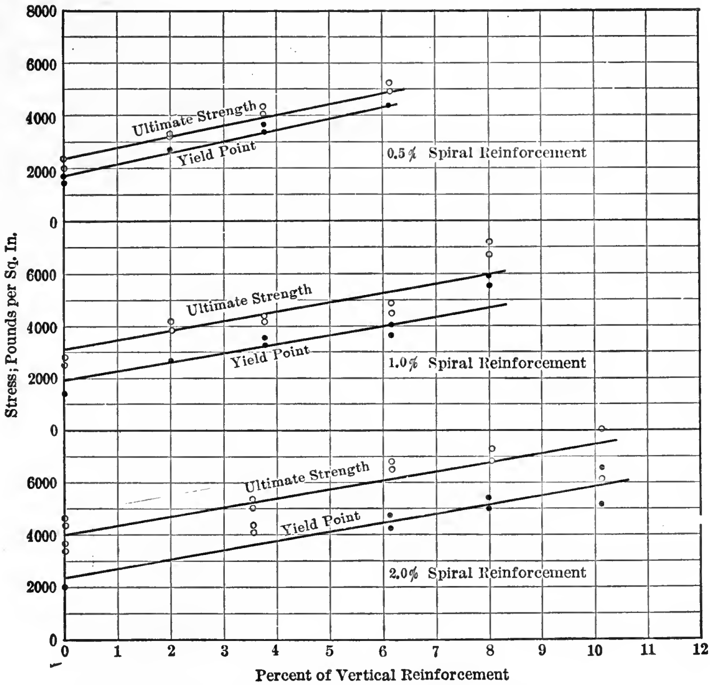


FIG. 5.—Tests of Reinforced Columns.

the longitudinal steel. The spiral steel is not included as its effect on the yield point is very small. The effect of the spiral steel on the ultimate strength is approximately equal to 500 to 1000 lbs/in² for each 1% of spiral steel, this increment to be added to the yield point strength as determined by the above formula. The effect of spiral steel on ultimate strength is therefore approximately in proportion to its yield point strength.

Table No. 29 gives results of tests made at the U. S. Bureau of Standards for a committee of the American Concrete Institute. The columns were made of 1 : 1½ : 3 mixture and

were 9 feet 4 inches long between enlarged heads and generally about 21 inches in diameter. They were therefore rather stocky columns, the ratio of height to diameter being about $5\frac{1}{2}$. The longitudinal reinforcement consisted of 1- to $1\frac{3}{4}$-inch round rods, and the hoop reinforcement of spirals of $\frac{3}{8}$ and $\frac{1}{2}$ inch in diameter. The elastic limit of the vertical rods was about 36,000 lbs/in^2 and of the spiral reinforcement about 62,000 lbs/in^2. The diameter of the spiral was about $19\frac{3}{4}$ inches and of the column from 1 to $1\frac{1}{2}$ inches larger. The calculations of strength per square inch are made on the total cross-section. The age of columns was from 100 to 125 days.

Table No. 29.

TESTS OF REINFORCED CONCRETE COLUMNS.

American Concrete Institute.*

No. of Tests.	Area of Cross-sec. Sq.in.	Per Cent Reinforcement.		Average Ultimate Strength, lbs/in².	Compressive Strength of 8″×16″ Cylinders, lbs/in².
		Longitudinal.	Spiral.		
2	343	0	0	2745	2646
2	320	.98	0	3750	3203
2	320	2.01	0	3690	3256
2	318	4.07	0	4535	2890
3	340	0	.94	4845	3012
3	345	.91	.94	4920	3009
3	344	1.83	.94	5235	2812
3	340	5.70	.94	6093	2850
2	343	.91	1.88	6258	2960

* Jour. Am. Concr. Inst., July 1915, Vol. 3, p. 424.

Fig. 3, giving typical curves from these tests, has already been referred to. Curve *A* represents a plain concrete column and curve *B* one with .94% spiral reinforcement. Curve *C* represents the same column as *A*, curve *D* a column with 2% longitudinal steel only and *E* a column with 1.83% longitudinal and .94% spiral.

Curves *F* and *G* show the effect of amount of spiral steel. Column *F* has .91% longitudinal and .94% spiral, and column

G has .91% longitudinal and 1.88% spiral, or double the amount of F.

In general it is seen that spiral steel adds to ultimate strength and especially to the deforming capacity of the column. It has little effect on the curve up to a point corresponding to the ultimate strength without the spiral. Column G has twice the amount of spiral reinforcement as F and its ultimate strength is greater, but up to a load of 1,200,000 lbs. the curves are the same. It would seem, therefore, that the actual working stress on G could hardly be placed higher than on F.

The ultimate strength added for each 1% of spiral steel was from 1200 to 1500 lbs/in², or about 3 to 4 times that furnished by the longitudinal steel, or about twice as much if account be taken of the difference in elastic limits.

Cast iron has been used to some extent in Europe as a material for longitudinal reinforcement. Hollow cast-iron cores are used with spiral reinforcement. Results of tests show the same general effect as in the case of steel. The hoops prevent the ultimate failure of the concrete until the iron core fails so that the ultimate strength of the combination is approximately equal to the sum of the ultimate strength of the concrete and that of the cast iron. Since cast iron has a high crushing strength this combination gives a column of high ultimate strength. It has a greater deformation than a steel-concrete column as the modulus of elasticity of cast iron is only about 10,000,000. A series of such columns tested by the U. S. Bureau of Standards with 5 to 7 inch cores constituting about 14% of longitudinal reinforcement, and with .6% spiral reinforcement gave an ultimate strength equivalent to about 4800 lbs. per sq. in. for the concrete and 45,000 lbs/in² for the iron. The ultimate strength of small specimen columns 5×24 inches was 72,700 lbs/in². European tests made by Emperger on iron of higher grade and with about 3% of spiral hooping gave results equivalent to as high as 8000 lbs/in² on the concrete and as high as 100,000 lbs/in² on the iron.*

* See article by L. J. Mensch, *Eng. and Contr.*, Feb. 28, 1917.

175. Conclusions as to Strength of Columns.—From the results of tests quoted we may draw the following conclusions: that the strength of plain concrete columns of 1 : 2 : 4 mixture at 60 days may be taken at from 1600 to 1800 lbs/in²; that very great gain in strength is shown for both plain and reinforced concrete by the use of richer mixtures; that the strength of columns reinforced with longitudinal rods only may be estimated in accordance with theory, but that the density and rigidity of the concrete itself is apt to be less in the reinforced than in the plain column, so that for small percentages of longitudinal reinforcement the gain in strength is small; that hooped columns, with or without longitudinal steel, show greatly increased deformation before rupture and a much higher ultimate strength than columns without such hooping, but the yield point is not materially affected; that more than 1% of hooping does not appear to be necessary to secure a desirable degree of toughness or ductility; that the addition of longitudinal steel to hooped columns increases greatly the elastic strength as well as the ultimate strength of the column.

If both great strength and toughness are to be secured, qualities usually to be desired, then both longitudinal and hoop reinforcement should be used.

176. Effect of Length of Column on Compressive Strength.—Comparing the results on plain concrete columns, Art. 170, with the tests on cubes, Chapter II, it is evident that the strength of the column is materially less. While there is thus a very considerable reduction of strength as compared to the cube, there appears to be little difference in the strength of columns of various lengths up to 15 to 20 diameters.

In the tests of Art. 172 the difference in averaging results upon the 8×8 inch columns and those on the 10×10 inch size is marked. But comparing results for each size among themselves there is little or no effect noticeable up to 25 diameters. Numbers 2 and 3 are reported as having failed by buckling, but these average practically the same as Nos. 1 and 4. Tests

by Bach on columns 4 feet long and columns 29 feet 6 inches long gave a strength for the latter of about 75% of the former. He proposes a long-column formula identical with that in eq. (3), Art. 169.* From the various results it would appear that no account need be taken of length of column below about 20 diameters, although caution should be used in accepting these results as conclusive. In the case of hooped columns the effect of buckling is evident for shorter lengths, inasmuch as this kind of column has a sufficient toughness to permit of considerable deformation before failure.

Considering the comparatively brittle nature of the column with longitudinal reinforcement only, it should not be used for columns of slender proportions. The banded or hooped columns is much more reliable for such work.

177. Working Stresses for Columns.—In determining the proper working stresses for columns it is necessary to consider the question mainly with reference to the stress in the concrete, for under ordinary working stresses in the concrete the stress in the steel will be relatively low. From the tests and discussion of the preceding articles it appears that with reference to the behavior of the concrete, columns may be divided into two classes: (1) columns reinforced with longitudinal reinforcement only, and (2) columns reinforced with hoops or bands and with or without longitudinal reinforcement. These types will be considered separately.

(1) *Columns Reinforced with Longitudinal Steel Only.*—In this form of column the concrete fails in a manner similar to the failure of an unreinforced column. When a load is reached which stresses the concrete to about the same value as in a plain concrete column, failure takes place suddenly and by shearing action. The ultimate strength of the entire column is, however, increased by the steel and approximately as theory would indicate. Considering the manner of failure and the lack of "toughness" in such a column the factor of safety

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* Zeit. Ver. Deutsch. Ing., 1913, p. 1969.

should be relatively large and determined on about the same basis as for a short column of plain concrete. From these considerations the Joint Committee recommends a working stress for the concrete of $22\frac{1}{2}\%$ of the crushing strength, or 450 lbs/in² for a 2000-lb concrete.

The working stress in the steel is a function of the working stress in the concrete and the ratio, n, of the moduli of elasticity of the two materials. The Joint Committee recommends a value of n of 15 for concrete having a strength between 800 and 2200 lbs/in², a value of 12 for concrete between 2200 and 2900 lbs/in², and 10 for concrete stronger than 2900 lbs/in².

(2) *Columns Reinforced with Hoops or Bands and with or without Longitudinal Steel.*—From the tests quoted, it is seen that in general the effect of hooping is to increase the "toughness" and the ultimate strength of the column. The elastic limit and rigidity of the column is not much affected. When used with longitudinal steel it keeps the concrete intact up to a degree of deformation that enables the longitudinal steel to be stressed to its elastic limit. It thus renders such reinforcement very effective.

Concerning the proper working stress for hooped columns, it would seem that this should be selected mainly with reference to the elastic limit, as in the case of structural steel; but the greater toughness of the hooped column, as compared to the other type, insures a much larger and more certain margin of safety, and hence the working stress may be made a greater proportion of its elastic limit strength than in the other case. The two types of columns may be compared to mild steel and cast iron; a much higher relative working stress may be used in the former than in the latter. chiefly because of its larger margin of safety against deformation beyond the elastic limit. This is of great importance, especially with respect to effects of unequal settlement, eccentric loading and secondary stresses.

The use of hooped columns without longitudinal steel is

not to be recommended. Hooping must be held securely in place during fabrication and to accomplish this with certainty a certain amount of longitudinal steel is required, and if this is properly placed it may be counted upon to carry its share of the load. The Joint Committee, in its revised report of 1916, gives no unit stresses for columns with hoop reinforcement only.

For hooped columns containing longitudinal reinforcement, the elastic limit of the column tends to approach a point corresponding to the elastic limit of the longitudinal steel, the exact effect depending upon the effectiveness of the hooping and the amount of longitudinal steel. If this effect were fully accomplished, the working stress might be placed as high as 15,000 lbs/in^2 on the steel, corresponding to a stress on the concrete of about 1000 lbs/in^2. This is beyond the normal elastic limit strength of the material, and is not to be recommended. The Joint Committee recommends for this type of column, containing not to exceed 4% of longitudinal steel and at least 1% of hoop reinforcement, a working stress 55% greater than for plain concrete; or about 700 lbs/in^2 for 2000-lb concrete. The Committee limits the recommendation to columns in which the ratio of length to diameter of hooped core is not more than 10.

In the determination of the strength of hooped columns, only the section within the hooping should be considered. The shell outside is of the same character as plain concrete and it is found to crack and split off at deformations corresponding to the ultimate strength of plain concrete. It is useful as fireproofing, but its limitations of deformation is another reason for not selecting too high values for the working stress on the core.

It should be said that the above treatment of the hooped column is quite different from that of Considère and of the French Commission on Reinforced Concrete. These authorities recommend that the hooping be counted upon directly to a much larger extent than the longitudinal rein-

forcement. The formula recommended by the French Commission is

$$f=f_c(1+15p+32p'),$$

in which f_c is the safe strength of plain concrete, taken at 28% of the ultimate strength in the form of cubes, p=ratio of longitudinal reinforcement, and p'=ratio of spiral reinforcement. It is also recommended that the maximum stress shall not exceed 0.6 of the ultimate strength of the concrete. These values are based chiefly on a consideration of ultimate strength.

(3) *Columns Reinforced by Structural Steel Column Units.*—Where a large amount of reinforcement is desired, certain advantages are gained by arranging it in the form of structural column units, such as four angles latticed together, which in themselves are capable of acting as columns.* The construction can be so arranged that the steel columns will carry the false work and dead load of two or more floors, thus enabling the placing of concrete to proceed simultaneously on several floors. In this way, also, some initial dead load stress can be applied to the steel of the column before the concrete of the column is placed, thus enabling higher steel stresses to be used. On the other hand, such steel is much more costly per pound than rods. Furthermore, the results of experiments show that the adhesion of concrete to steel where the latter presents broad flat surfaces is not good, and the presence of numerous lattice bars hinders the production of a dense homogeneous concrete. The resulting column is therefore likely to be less of a monolithic character than one in which the reinforcement consists of small rods. In order that the concrete may be counted upon in such a column, it should be well enclosed either by the structural form itself or by means of bands or hooping. All concrete not so enclosed can be considered only as fire-proofing.

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* For a good example of such a design, see paper by Wm. H. Burr on "The Reinforced Concrete Work of the McGraw Building," Trans. Am. Soc. C. E., Vol. 60, 1908.

Where designed in accordance with these principles, and the steel and concrete receive their load simultaneously, the working stresses may be taken about the same as for the second class of columns here discussed. If, however, a partial load is applied to the steel before the concrete is placed, such initial stress need not be counted, excepting that the total stress in the steel should not exceed the usual working stress for steel columns of about 16,000 lbs/in^2. Where the amount of steel becomes very large the relative value of the concrete becomes more uncertain and its consideration as an element of strength is of doubtful wisdom and unsupported by experimental evidence.

178. Column Details.—In the construction of columns great care should be exercised to place and hold the steel in its proper position and to secure sound work. In this respect poor workmanship is more serious perhaps than in any other structural form. Eccentricity of steel or uneven quality of concrete not only causes weakness at the section in question, but also results in eccentricity of load and lateral deflections. Reinforcing rods must be arranged concentrically and held securely in place until the concrete is set. This is usually accomplished by wiring or banding the rods together at intervals of a foot or so, but such banding cannot be considered as hooping in the sense usually employed. Where hooping is used as reinforcement it may consist of wire spirally wound or otherwise, or of separate bands of welded or riveted steel. To be effective such hooping should be spaced relatively close, so as to serve to confine the concrete within the cylinder formed by the hooping and to effect the "toughening" assumed in the previous discussion. Such spacing should not exceed one-sixth the diameter of the enclosed column core. A total amount of hooping or banding at least equal to 1% of the enclosed volume should be used.

In the case of hooped columns or columns in which lateral reinforcing members are used, such as lacing on structural units, special care should be taken to secure as dense concrete as possible, and to reduce the settlement of the material to a

minimum. Any settlement tends to create vacant spaces or porous material underneath the reinforcement. In splicing columns large rods or structural shapes should be accurately fitted and well spliced; small rods may be spliced at floor levels by overlapping a sufficient distance to develop the requisite bond strength. At the base of a column large rods or shapes should rest upon suitable base plates in the foundation concrete.

CHAPTER VIII.

ANALYSIS OF FLAT SLABS.

179. In the preceding chapters, we have dealt with beams on the assumption that they were individual units carrying definite loads, and that the stresses on any transverse section were uniformly distributed in a lateral direction,—that is, were of equal intensity at all points equi-distant from the neutral axis. This treatment holds good, of course, for beams or slabs of indefinite width, supported along lines transverse to the axis, provided the loading is uniformly distributed over the entire width of the beam. In such a case it is convenient and satisfactory to analyze a strip of beam one foot wide. But there are many cases, both in building and in bridge construction, of broad beams or slabs in which conditions of support or loading are such that they cannot be analyzed by the above method. The important cases of this kind are the following:

A. Broad beams or slabs, supported along two sides only, but loaded with concentrated loads, such as bridge floors supporting concentrated loads.
B. Rectangular slabs supported on four sides.
C. Footings, in which a flat slab supports one or more columns.
D. Floor slabs supported directly on columns, this arrangement constituting the so-called "flat slab" construction.

In all these cases the structure is a statically indeterminate one, that is to say, the stresses cannot be determined in detail from the principle of statics alone; relative deflections or deformations must be considered. An exact analysis of these problems is impracticable, if not impossible; they can be solved only approximately. The several cases mentioned above will be considered in the order given.

A. SLAB BEAMS SUPPORTED ALONG TWO SIDES.

180. General Conditions. — Where concentrated loads are applied to beams of great width, it is necessary to determine approximately the manner in which such loads are distributed laterally over the beam. A common example is a bridge floor consisting of a concrete slab resting on steel or concrete beams, the beams running either parallel or transverse to the axis of the roadway. In either case the stresses caused by a concentrated load, such as a road roller, involve a determination of the extent to which such a concentration is distributed laterally over the beam. The problem may be considered in two parts, (1) the lateral distribution of load over a beam of given width, and (2) the determination of the "effective" width of a beam of indefinite or very great width. Only a rough theoretical treatment of this problem can be given, but the results of analysis may serve as a basis for reasonable rules of practice, and for rational interpretation of experimental data, which are as yet very limited.

181. Lateral Distribution of Concentrated Loads on Slab Beams.—Fig. 1 represents a plan view of a slab beam of length

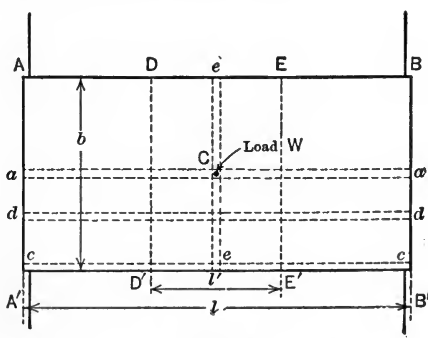


FIG. 1.

l and width b, simply supported at AA' and BB'. The principal reinforcement is longitudinal, but the beam is also generally reinforced laterally to some extent. A concentrated load W is applied at the center. The problem is to determine the relative

proportion of this load carried by a central element *aa* and by any other element *dd* or *cc*.

In general, the relative loads carried by the various longitudinal elements will be proportional to their deflections, and it is evident that the point *e* on *cc* will not deflect as far as the point *C* under the load. The difference in these deflections will be measured by the upward deflection of point *e* with respect to *C*, due to the lateral transfer of load. This upward deflection will now be estimated and compared to the downward deflection of the entire beam along the line *ee*.

It is obvious that the lateral transfer of load from *C* toward the edges of the beam is not accomplished entirely by a narrow strip *ce* under the load, but to a variable extent throughout the entire length of the beam. For a certain length l', it may be assumed that the lateral transfer of load will be practically uniform, and it will be sufficiently exact for our purpose to assume the transfer of load to be concentrated over this length l'. This length may safely be taken at $\frac{1}{4}$ or $\frac{1}{3}$ of the length l. It will then be assumed that one-half the load W is transferred laterally in each direction over a length of $l'=\frac{1}{4}l$. It will also be assumed that the transverse strips *ee*, etc., are so rigid that they transfer the load equally to each longitudinal strip *cc*, *dd*, etc. Each half of the beam $DED'E'$ acts, then, as a cantilever beam loaded with a load W at the center and a uniform upward load equal to $W/2$ on each half, causing a small upward deflection of the edges DE and $D'E'$ with reference to the center C. The downward deflection of the entire beam on the line *ee* will be a maximum at C and a minimum at e; its average value will be about the same as if the load W were uniformly distributed along the line *ee*. It will be so calculated. Let

Δ = downward deflection of beam center, assuming the load W uniformly distributed along the line *ee*;

Δ' = upward deflection of point e relative to C;

I = moment of inertia of a longitudinal strip of beam, *aa*, one unit wide.

It will be sufficiently accurate for present purposes to assume that the moment of inertia of a transverse unit strip *ee*, is the same as that of a longitudinal strip. There will be differences in the amount of steel, but the concrete will be the same, and, under working conditions, the differences in steel will have little effect on deflections. Hence, the total moment of inertia of the longitudinal beam will be bI and of the transverse beam $DED'E'$ will be $l'I$.

Under the above assumptions, the deflections will be as follows:

$$\Delta = \frac{Wl^3}{48EIb}, \qquad (1)$$

$$\Delta' = \frac{\frac{W}{2}\left(\frac{b}{2}\right)^3}{8EIl'} = \frac{Wb^3}{32EIl}. \qquad (2)$$

Comparing Δ' with Δ, we have

$$\frac{\Delta'}{\Delta} = 1.5\left(\frac{b}{l}\right)^4. \qquad (3)$$

That is, the ratio of deflections is approximately proportional to the fourth power of the ratio b/l.

If, for example, $b/l = \frac{1}{3}$, then $\Delta'/\Delta = 1.5/81 = .02$; that is to say, the edges of such a beam will be deflected upwards only about 2% as much as the *average* downward deflection, or about 3% as much as the *maximum*. The load W is therefore distributed transversely practically uniformly on a beam of such proportions.

182. Effective Width of Beams of Indefinite Width.—For slabs whose width is indefinite, it is desirable to estimate the width of the portion over which the distribution may be considered practically unform. For such beams, it would be reasonable to place a limit of variation of deflection at,

say, 20% from the average. Then placing $\Delta'/\Delta=.2$, we have,

$$\left(\frac{b}{l}\right)^4=\frac{1}{7.5},$$

from which

$$b=0.6l. \qquad (4)$$

If the length l' be assumed at $l/3$, as would appear reasonable from the results obtained above, the result would be

$$\Delta'/\Delta=1.12\left(\frac{b}{l}\right)^4,$$

and for $\Delta'/\Delta=.2$,

$$b=.65l.$$

It should be noted that in the case of such a beam as here considered, the edges of the strip of width b are not free to bend up as cantilever beams, but are restrained by the portion outside the width considered, hence the ratio of Δ' to Δ is doubtless considerably less than here assumed.

From this analysis, it would appear that it is reasonable to assume an even distribution of load over a width of about two-thirds the length of the beam. Where the concentrated load itself is distributed over a considerable width, such width may be added to the width b as above determined.

183. Transverse Bending Moments in Slab Beams.—The average bending moment in a longitudinal strip one unit wide will be

$$M=\frac{Wl}{4b}. \qquad (1)$$

The bending moment M' in a transverse strip one unit wide, assuming $l'=l/3$, and acting as assumed in Art. 181, will be

$$M'=\frac{W}{2l'}\times\frac{b}{4}=\frac{3}{8}\frac{Wb}{l}. \qquad (2)$$

Making $b = .6l$ gives the following values:

$$M = .42W, \qquad (3)$$

$$M' = .22W. \qquad (4)$$

Equations (3) and (4) indicate that the transverse bending moment, under the conditions assumed, is approximately one-half the longitudinal moment. Tests of slabs appear to show that under working loads the transverse moment is considerably less than this, good results being obtained on beams with little or no transverse reinforcement. An amount of transverse reinforcement equal to $\frac{1}{4}$ to $\frac{1}{3}$ the longitudinal is ample in any case. For loads approaching the ultimate, the transverse moments will become relatively greater, as the deflection will depend more and more on the steel, and less on the concrete. Final failure from concentrated loads generally occurs by the load punching through the slab, a combination of punching shear and diagonal tension.

184. Tests of Slab Beams.—Tests have been made by Professor C. T. Morris * on slabs 4 inches and 7 inches thick, spans from $3\frac{1}{2}$ to 7 feet and widths from 1 foot to 7 feet. The longitudinal reinforcement amounted to 1.04% and the transverse reinforcement .2% to .78%. His general conclusions were as follows:

1. The effective width is affected very little by the percentage of transverse reinforcement.

2. The effective width decreases somewhat as the load increases.

3. The effective width in percentage of the span decreases as the span increases.

4. The following formula will give a safe value of effective width where the total width of slab is greater than $1.35l + 4$ feet.

$$b = 0.6l + 1.7,$$

where b = effective width in feet, and

l = span-length in feet.

* Bulletin No. 28, Highway Department, Ohio, 1915.

The "effective width" is that width over which the entire concentrated load may be assumed as distributed so that the resulting calculated stresses will be about equal to the maximum actual stresses.

The Office of Public Roads, U. S. Department of Agriculture, conducted a series of tests on a slab beam 12 inches thick of 16-foot span length, and a width of 32 feet. For a single concentrated load the effective width was found to be about 11 feet, for loads producing concrete stresses about equal to ordinary working stresses. The slab had 0.75% longitudinal reinforcement, but no transverse reinforcement. This value of the effective width is about $.7l$.*

185. Distribution of Load from Continuous Slabs to Supporting Joists.—Where a continuous slab is supported by several beams or joists, it becomes necessary to estimate the maximum load on any one joist caused by a concentrated load on the slab. It is a problem depending upon the relative flexibility of the slab and the supporting beams. It will be advantageous to derive certain theoretical formulas covering two simple cases. These will indicate the probable range of values and the effects of the various elements of the problem.

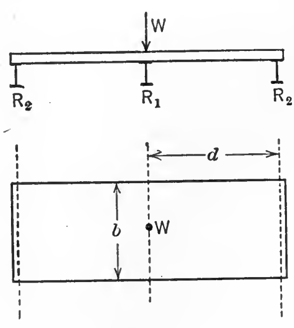


FIG. 2.

We will assume, first, a slab of width b resting upon three longitudinal supporting beams, Fig. 2, and supporting a concentrated load W placed at the center. The problem is to determine the proportion of the load W carried by the center beam, or the values of the reactions R_1 and R_2.

Let E = modulus of elasticity of slab;

I = moment of inertia of slab;

K = coefficient of flexibility of supporting beam = de-

* *Engineering Record*, Vol. 71, 1915, p. 26.

flection in inches for a load of one pound applied where the slab is placed. This can readily be computed for any given length and section of beam;

d = spacing of beams in inches;

a = a constant $= \frac{KEI}{d^3}$.

Then, by placing the deflection of the slab at the center equal to the difference in deflection of the center and outside beams, we can solve for the value of R_1, obtaining

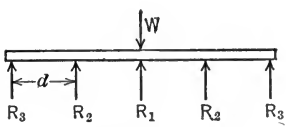


FIG. 3.

$$R_1 = \frac{1+3a}{1+9a} \times W. \qquad (1)$$

Then

$$R_2 = \frac{W - R_1}{2}. \qquad (2)$$

If five supports are assumed instead of three, a similar analysis gives the following results (Fig. 3):

$$R_1 = \frac{7+108a+36a^2}{7+204a+180b^2} \times W, \qquad (3)$$

$$R_2 = \frac{11+6a}{16+24a}(W - R_1), \qquad (4)$$

$$R_3 = \frac{W - R_1 - 2R_2}{2}. \qquad (5)$$

To show how the values of R_1 vary with values of the constant a, the curves of this function for values of a from .05 to 3.0 are plotted in Fig. 4. It will be seen that for values of a less than 1.0, the value of R_1 is about the same, whether three supports or five supports are used. Small values of a correspond to rigid or widely spaced supports and thin or flexible slabs; large values of a, to stiff slabs and flexible supports.

Ordinary values of the constant a may be estimated as follows:

For a slab of indefinite width, the effective width b may, in view of the analysis of Art. 182, be taken about equal to d.

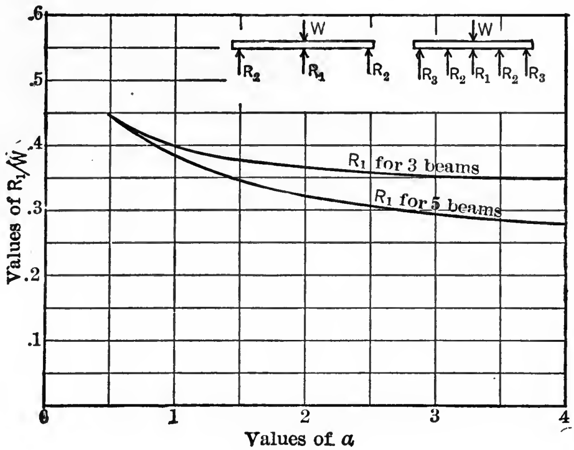


FIG. 4.

The value of I is then equal to $\frac{1}{12}bh^3$. Assume $E=3,000,000$. The quantity K is the deflection of the supporting beam for a one-pound load distributed over the length b. For a center load, the deflection is $\Delta=\frac{1}{48}\frac{Wl^3}{E_bI_b}$, and for a uniformly distributed load it is $\frac{5}{384}\frac{Wl^3}{E_bI_b}$, where $W=$ total load on beam, and E_b and I_b refer to the beam. As the load is only partially distributed, the value of K or the deflection for a load of one pound may be taken at about $\frac{1}{60}\frac{l^3}{E_bI_b}$.

Assuming a beam of steel, $E_b=30,000,000$. Then if $b=d$, we have

$$a=\frac{KEI}{d^3}=\frac{l^3}{60E_bI_b}\times\frac{E\cdot\frac{1}{12}bh^3}{d^3}$$

$$=\frac{l^3d}{7200I_b}\times\frac{h^3}{d^3}. \qquad (6)$$

The value of $\frac{l^3 d}{7200 I_b}$ will ordinarily range from 150 to 300, hence, approximately

$$a = (150 \text{ to } 300)\frac{h^3}{d^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

We have, then, for a very thin slab such that $\frac{h}{d} = \frac{1}{10}$, $a = .15$ to $.30$, and from Fig. 4, $R_1 =$ from $.5W$ to $.55W$. For $h/d = \frac{1}{7}$, $a = .45$ to $.9$, and $R_1 = .45W$ to $.50W$, and for $h/d = \frac{1}{5}$, $a = 1.2$ to 2.4, and $R_1 = .32W$ to $.38W$.

As the ratio of thickness to stringer spacing will generally range from $\frac{1}{5}$ to $\frac{1}{7}$, the value of R_1 may, ordinarily, be taken at about $.40W$.

The test by Professor C. T. Morris for the Ohio State Highway Department already mentioned included tests on the distribution of load to several beams. Slabs, 6, 7, and 8 inches thick, supported on three rows of 10-inch, 25-pound I-beams, gave a load on the center beam from 32% to 48% of the total load. The value of a was in this case about $130\frac{h^3}{d^3}$ and $\frac{h}{d}$ ranged from $\frac{1}{5}$ to $\frac{1}{7}$.

186. Bending Moment in Slabs Supported on Several Beams. —The bending moment in a slab resting on several supports will depend upon the relative values of the several reactions and the spacing of the beams. For three supports and $R_1 = .4W$, $M = .3Wd$. For five supports and $R_1 = .4W$, $M = .36Wd$, and for $R_1 = .35W$, $M = .40Wd$. For four supports, load placed at the center, the value of M ranges from $.35Wd$ for $a = .5$ to $.44Wd$ for $a = 2$.

The above values relate to the effect of a load concentrated along a narrow line along the center parallel to the beams. Considering the fact that any heavy load is distributed over a considerable width, it would appear that a value of $M = .35Wd$ would be ample, as representing the total moment in the portion of the slab considered. For heavy loads, distributed laterally over a considerable distance, such as loads from road rollers,

etc., the moment as above found may be divided by the number of stringers over which the load extends. Or, if c=width of load, then $M=.35Wd\Big/\frac{c+d}{d}$. If $c=d$, then $M=.35Wd/2$, etc.

The bending moment per foot of width of beam will be equal to the total moment as above determined, divided by the effective width. Taking this as equal to d, we find the moment per foot of width to be equal to $.35W$ or $.35W\Big/\frac{c+d}{d}$, as the case may be. Thus, the moment per foot, in the case of a narrow, concentrated load, is independent of the stringer spacing. It would follow that for such a case, the thickness of slab (neglecting effect of dead load) is also independent of stringer spacing. This appears to be an unreasonable result but, considering the greater distribution of load in a longitudinal direction which results from a wide stringer spacing, it can be seen that within reasonable limits the stringer spacing will have little effect on the bending moment per foot of slab for a concentrated load.

B. RECTANGULAR SLABS SUPPORTED ON FOUR SIDES.

187. General Conditions.—If a panel between beams is square, or nearly so, the slab may advantageously be reinforced in both directions. The exact analysis of stresses in such a case is difficult, if not impossible, as the effect of the more or less flexible supports is especially important, and the problem is otherwise difficult of exact treatment. The following solution for uniform loading will serve to show, approximately, the relation of the loads carried by the two systems of reinforcement. The results are certainly safe, and do not vary much from rules of practice, but point to a somewhat more economical use of material.

188. Square Slabs.—In this case the reinforcement should be of equal amount in the two directions. It may be calculated on the assumption that one-half the load is carried by each system of reinforcement. The concrete is proportioned for

only one system, or one-half the load, as the compressive stresses due to the two systems are at right angles to each other, and it is known that the stresses in one direction do not weaken the concrete with respect to stresses in the other direction. The loading on each system is usually assumed to be uniformly distributed, resulting in an equal spacing of rods throughout the beam. This assumption is, however, far from the truth, and while giving safe results a more exact analysis of the problem will show that the spacing of the rods near the edges can be made considerably greater than that in the centre.

In Fig. 5, *ABCD* represents a square slab supported on all sides and loaded with a uniform load w per unit area. Consider

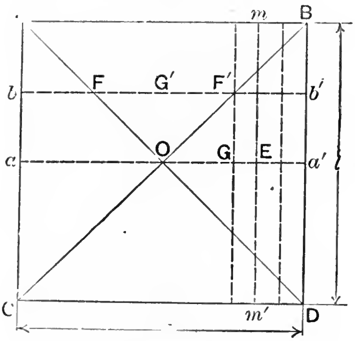


FIG. 5.

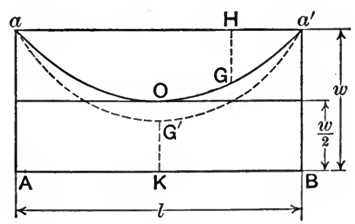


FIG. 6.

the relative amounts of load carried by the system parallel to aa' and the system parallel to mm'. At the center O, and at all points on the diagonal lines AD and CB, it follows from symmetry that the loading is equally distributed on the two systems and is equal to $w/2$. At point E the proportion of the load carried by the system aa' will be much greater than that carried by the system mm', since for given loads the beam element aa' will deflect much less at point E than will the element mm'. In general, therefore, as we approach the support BD the proportion of load carried by the system aa' increases, reaching a value of w at the extreme end a'. The distribution of load on aa' may then be roughly represented

in Fig. 6 by the ordinates from AB to the curved line aOa'. Consider now the load along an element bb'. At points F and F' the load will be $w/2$, at point G it will be less than $w/2$, being the same as the load on the system mm' at G. It is given in Fig. 6 by the ordinate GH from Oa' to the line aa'. At the end b' the load will be w. The curve of distribution will then be somewhat as represented by the line $aG'a'$ in Fig. 6, in which $G'K=GH$.

Assuming the curve aOa' to be a parabola, it is found that the center bending moment on the line aa' for a beam one unit wide will be $\frac{7}{48}(w/2)l^2$ instead of $\frac{6}{48}(w/2)l^2$, as results from the assumption of uniform distribution. The spacing of the rods at the center may then be determined on this basis. At points intermediate between the center and the edge, the rods may well be spaced so that the number per foot will vary from the required number at the center to zero at the edge, following the law of the parabola. If N represents the total number required on the ordinary assumption of equal spacing, then $\frac{7}{6}N\times\frac{2}{3}$, or $\frac{7}{9}N$, will represent the number when spaced as here calculated. Practically as good results will be secured if the rods are spaced uniformly at the usual spacing, determined by the formula $M=\frac{1}{8}(w/2)l^2$, or the center half of the slab, then gradually reduce the number per foot to the edge of the slab, using one-half as many rods for the remaining two quarters. The total number used would then be $\frac{3}{4}N$ instead of $\frac{7}{9}N$ as above determined, but the strength would be ample.

189. Continuous Slabs.—If the slabs are continuous then the values above given for moment may be reduced by using a factor $\frac{1}{12}$ instead of $\frac{1}{8}$; as is commonly done in the case of ordinary beams. That is, the values of the moments may be taken at two-thirds the values given in Art. 188.

190. Oblong Slabs.—As a slab becomes oblong in form, the relative amount of load carried by the longitudinal system becomes rapidly less. Fig. 7 represents an oblong slab of length l and breadth b. Consider a central strip one foot wide along the line aa' and also along the line mm'. Suppose the

rods to be spaced equally in the two directions so that the moments of inertia of the strips are equal. Let $w_1 =$ load per foot on the strip mm' and $w_2 =$ load per foot on aa'. The deflection of a beam uniformly loaded is proportional to wl^4, hence, since the deflections of the two beams are equal, we have $w_1 l^4 = w_2 b^4$ or $w_1 : w_2 = b^4 : l^4$. That is, the amount of the load carried (per square foot) by the two systems is inversely proportional to the fourth power of the respective dimensions. For points near the ends of the slab the proportion carried by the longitudinal system will be relatively large, but in any case the longitudinal rods will be much understressed.

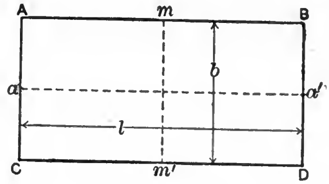


FIG. 7.

In accordance with this theory the proportions of the total load carried by the two systems for various ratios of $l : b$ are as follows:

Ratio $l : b$	1.0	1.1	1.2	1.3	1.4	1.5
Proportion of load carried by transverse system	.50	.59	.67	.75	.80	.83
Proportion of load carried by longitudinal system	.50	.41	.33	.25	.20	.17

The coefficients above given assume that each system is designed to carry its full proportion as a uniformly distributed load. Using these coefficients in the usual bending moment formula, the total load will be $wbl = W$. The average transverse moment per unit width will be $\frac{1}{8}kW\frac{b}{l}$, and the average longitudinal moment will be $\frac{1}{8}kW\frac{l}{b}$, where k is the coefficient above given

For example, in the case of a slab whose length is 1.5 times its breadth, the average moment per unit width on the transverse system will be $\frac{.83W}{8} \times \frac{1}{1.5} = .069W$, and the average

moment on the longitudinal system will be $\frac{.17W}{8}\times 1.5 = .032W$.

For beams whose length does not exceed 1.5 times their width the spacing of rods may well be reduced in the outer quarters in the same manner as recommended for square slabs.

191. Exact Methods of Analysis.—The mathematical analysis of homogeneous plates supported on unyielding supports has been worked out by several investigators, but the application of these methods is very laborious. Considering the nature of reinforced concrete and other factors involved which tend to modify the assumptions made in the mathematical analysis, the results obtained can be considered as only a rough approximation and as a check on simpler methods of analysis.

Probably the most practical solution is that by Mr. H. Hencky * who has worked out numerical coefficients and curves of moments in rectangular plates for both uniform and concentrated loads. The value of Poisson's ratio is taken at $\frac{1}{3}$. His results for rectangular plates supported at the edges and uniformly loaded are as follows:

Ratio $l : b$	1.0	1.2	1.4	1.6	1.8	2.0
Proportion of load carried by transverse system	.30	.38	.44	.50	.54	.60
Proportion of load carried by longitudinal system	.30	.25	.20	.15	.12	.11

It will be noted that the sum of the coefficients does not equal unity. While the total load must of course be carried, the above coefficients are intended to be used merely in calculating the average moment in the usual formula for uniformly loaded simple beams.

For square slabs fixed at the edges, Hencky finds the moment at the center to be about one-half the value for free ends, and the negative moment at the center of the edge practically the same as the center moment for free ends.

* Der Spannungszustand in Rechteckigen Platten. Berlin, 1913.

192. Distribution of Slab Loads to Beams.—Where the floor-slab is reinforced in one direction only the load will practically all be transmitted to the corresponding beams, but at the ends of the panels a small part will be transferred directly to the girder. This may be neglected in the calculations. In the case of reinforcement in two directions, unless the panel is nearly square, the load may still be assumed as all transferred to the side beams. If the panels are square, or nearly so, the distribution may be assumed in accordance with the discussion of Art. 188. Thus the load brought to point a'

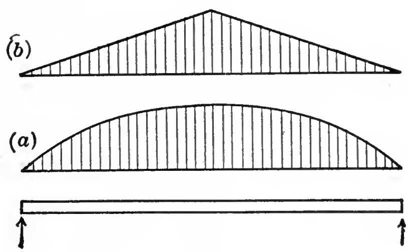


FIG. 8.

(Fig. 6) will be one-half of the area below the curve aOa', and the load brought to b' will be one-half the area below the curve $aG'a'$, etc. The distribution along the beam will then follow some such law as represented by the shaded area in Fig. 8 (a), the total load on one beam being necessarily $\frac{1}{4}wl^2$, where w = floor-load per square foot. It will be sufficiently accurate to assume this curve a parabola. The center bending moment in the beam, assumed as a simple beam, will then be equal to

$$M=\frac{5}{128}wl^3.$$

A distribution of load as represented in Fig. 8 (b), as is sometimes assumed, gives a center moment equal to $\frac{1}{24}wl^3$, a value about 7% higher than the above. A uniform distribution gives a moment equal to $\frac{1}{32}wl^3$, a value 20% lower.

C. CIRCULAR SLABS SUPPORTED AT THE CENTER; FLAT SLAB FOOTINGS.

193. The exact determination of stresses in flat slab floor systems and footings involves very complex analytical processes and is impracticable. These structures must be designed on the basis of approximate calculations and estimates regarding the manner of distribution of bending stresses along certain significant lines. It will assist in making these estimates to consider the theoretical analysis of a flat circular plate supported at the center.

194. Circumferential and Radial Bending Moments.—Let *AB*, Fig. 9, represent a circular plate loaded on the upper surface and supported over a certain area *C* at the center by means of a column or pier. The plate will be subjected to bending stresses which will deflect it into a sort of umbrella-shaped figure. Any element *a* will be subjected to moment stresses in both directions *FG* and *DE*, each producing tension in the upper part and compression in the lower part. The bending moment producing the stresses in the direction *FG* at *a* is called a *circumferential* moment, and that in the direction *DE* is called a *radial* moment. The values of the circumferential and radial moments at any point are given herewith for two cases of loading and two methods of support. The analysis is based upon general formulas * for a homogeneous circular plate of uniform thickness. The value of Poisson's ratio has been taken at $\frac{1}{10}$. The results are given in the diagrams of Figs. 10 and 11.

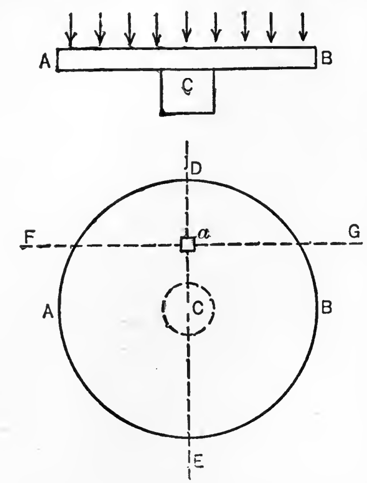
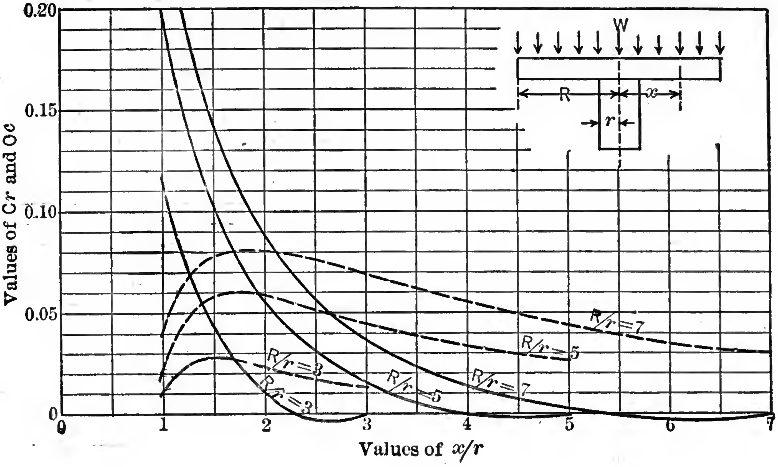


FIG. 9.

* Dr. H. T. Eddy, *Year Book, University of Minnesota*, 1899, or Morley's *Strength of Materials*, Chap. XIII.



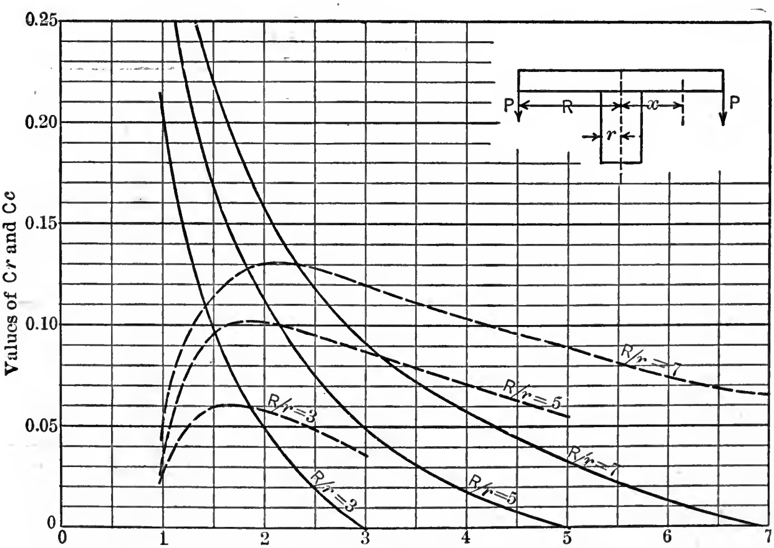
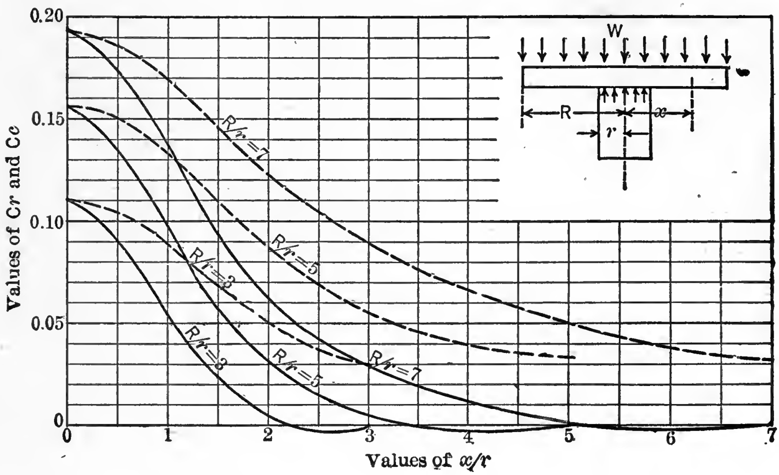


FIG. 10.—Moment Coefficients for Circular Slabs.



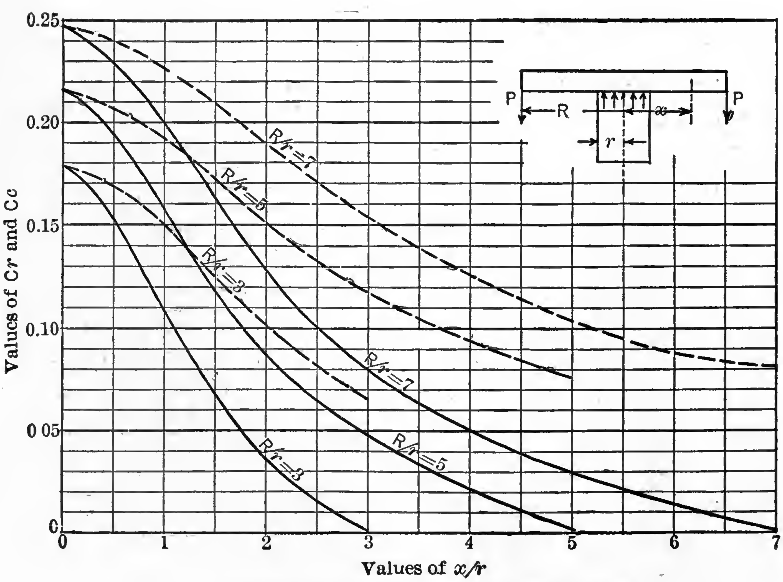


FIG. 11.—Moment Coefficients for Circular Slabs.

In the case given by Fig. 10 (Case A) the plate and support are assumed to be joined so rigidly that the tangent to the plate is horizontal at the support; that is, the support is assumed to be perfectly rigid. In Fig. 11 (Case B) the support is assumed to be such that the reaction is uniformly distributed over the area. In both cases, coefficients are given for two kinds of loading; (1) a load of W distributed uniformly over the surface, and (2) a peripheral load P distributed uniformly around the circumference. In all cases the bending moments are the moments per unit width of section.

For the uniformly distributed load W

$$\left.\begin{aligned} M_r &= C_r W \\ M_c &= C_c W \end{aligned}\right\}, \qquad \qquad (1)$$

and for the peripheral load P

$$\left.\begin{aligned} M_r &= C_r P \\ M_c &= C_c P \end{aligned}\right\}, \qquad \qquad (2)$$

in which M_r and M_c are, respectively, the radial and circumferential bending moments in foot-pounds per foot of width, or inch-pounds per inch, and C_r and C_c are the corresponding coefficients given by the solid and dotted lines respectively in the diagrams. Ratios of radius R of the plate to radius r of the support, have been taken at values from 3 to 7.

Comparing the two kinds of supports, uniform load W, it is seen from Fig. 10 that the radial moments are the maximum at the edge of the support and fall off very rapidly for points only a short distance therefrom, reaching zero at points within the plate. The circumferential moments increase for some distance and then fall off to a lower value at the edge. In Fig. 11 the moments are equal at the center and the radial moments fall off more rapidly than the circumferential, the latter being of considerable magnitude at the edge. In practice neither of the conditions (A) or (B) are exactly realized. Where the slab is thin and flexible and is built into a continuous

column of large size, the real condition will doubtless approach (A) nearer than (B), but the stresses and deformations in the outer portion of the column will always allow some deflection of the slab at the columns so as to materially reduce the radial moments below the values given in (A). Where the slab is thick and heavily loaded, and the column or support does not extend through the slab, as in a spread footing, the condition will approach nearer to (B).

Examples.—(1) A circular plate 10 feet in diameter is supported on a column 2 feet in diameter. The plate supports a load of 150 lbs./ft.² over its surface and a load of 500 lbs./ft. around its circumference. Required the radial and circumferential bending moments at the edge of the support.

$$W = 150 \times \pi \times 5^2 = 11{,}800 \text{ lbs.}$$

$$P = 500 \times 2\pi \times 5 = 15{,}700 \text{ lbs.}$$

$$R/r = 5.$$

CASE (A). Assume the plate supported rigidly by the column. Use Fig. 10. Then for load W, $C_r = .20$ and $C_c = .02$. For load P, $C_r = .30$, $C_c = .032$.

$$M_r = 11{,}800 \times .20 + 15{,}700 \times .30 = 2360 + 4710 = 7070 \text{ ft.-lbs. per ft.}$$

$$M_c = 11{,}800 \times .02 + 15{,}700 \times .032 = 240 + 500 = 740 \text{ ft.-lbs. per ft.}$$

CASE (B). Assume the reaction of the column to be uniformly distributed, Fig. 11.

For load W, $C_r = .10$, $C_c = .13$. For P, $C_r = .16$, $C_c = .19$.

$$M_r = 11{,}800 \times .10 + 15{,}700 \times .16 = 1180 + 2500 = 3680 \text{ ft.-lbs. per ft.}$$

$$M_c = 11{,}800 \times .13 + 15{,}700 \times .19 = 1530 + 3000 = 4530 \text{ ft.-lbs. per ft.}$$

(2) Calculate the moments in the plate of Ex. 1 at a point 2 feet from the center, $x/r = 2$.

CASE (A). For load W, $C_r = .056$. $C_c = .059$. For load P, $C_r = .112$, $C_c = .102$.

$$M_r = 11{,}800 \times .056 + 15{,}700 \times .112 = 660 + 1760 = 2420 \text{ ft.-lbs. per ft.}$$

$$M_c = 11{,}800 \times .059 + 15{,}700 \times .102 = 700 + 1600 = 2300 \text{ ft.-lbs. per ft.}$$

CASE (B). For load W, $C_r = .032$, $C_c = .086$. For load P, $C_r = .087$, $C_c = .15$.

$$M_r = 11{,}800 \times .032 + 15{,}700 \times .087 = 380 + 1370 = 1750 \text{ ft.-lbs. per ft.}$$

$$M_c = 11{,}800 \times .086 + 15{,}700 \times .15 = 1010 + 2350 = 3360 \text{ ft.-lbs. per ft.}$$

195. Total and Average Bending Moment Stresses on a Central Section.—For purposes of practical design it will be useful to consider the total and average bending moment stresses on a central section through the slab. Where the slab is built monolithically with the column, the usual case, the section to be considered is along the line *ABCDE*, Fig. 12. The stresses acting on this section are bending stresses, circumferential along *AB* and *DE* and radial along *BCD*, and shearing stresses on the portion *BCD*.

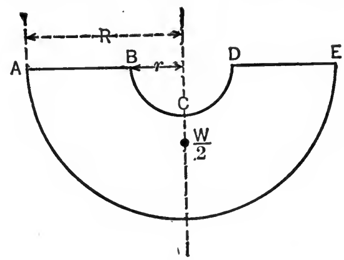


FIG. 12.

If the radial moments along *BCD* be resolved into components normal and parallel to *AE*, the normal components will be equivalent to a uniform bending moment along the projection *BD* of an intensity equal to that at *C*. For equilibrium the sum of the stress moments normal to *AE* must equal the bending moment due to the external loads and shears taken about the same axis. This bending moment is readily found in any given case, whence the total stress moment on the section *ABCDE* becomes known. Its distribution can then be approximately determined by the analysis of Art. 194. The total bending moment for the case of a circular plate is determined as follows:

For a load w per unit area on the plate, the centroid of pressure on the half area *ABCDE* is at a distance from the center equal to $\frac{4}{3\pi}\cdot\frac{R^3-r^3}{R^2-r^2}$. The half load itself $=W/2=\frac{1}{2}\pi w(R^2-r^2)$. The centroid of the shears on *BCD*, assumed uniformly distributed, is at a distance from the center equal to $2r/\pi$, and the total shear $=W/2$. Hence the moment of $W/2$ is equal to

$$M_w=\frac{W}{2}\left(\frac{4}{3\pi}\frac{R^3-r^3}{R^2-r^2}-\frac{2}{\pi}r\right). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a load P distributed uniformly around the circumference, the centroid of one-half the load $P/2$ is at a distance $2R/\pi$ from the center, and the reaction along BCD is $2r/\pi$ from the center. Hence the moment due to $P/2$ is

$$M_p=\frac{P}{2}\cdot\frac{2}{\pi}(R-r)=\frac{P}{\pi}(R-r). \quad . \quad . \quad . \quad . \quad (2)$$

Equations (1) and (2) give the total bending moments on the section $ABCDE$. Their distribution may be approximately determined from the diagrams of Figs. 10 and 11. This will be illustrated in the following Article.

196. Application of Preceding Analysis.—*Examples.* (1) Required the total bending moment on section $ABCDE$, normal to AE, for the slab of the example in Art. 194. The load per square foot on the surface $=w=150$ pounds and the peripheral load per lineal foot $=p=500$ pounds. $R=5$ feet, $r=1$ foot. Assume the slab supported uniformly over the column area; Case (B). The load $\frac{W}{2}$ on the half slab $AE=\frac{1}{2}\pi w(R^2-r^2)=$ 5600 pounds. Hence $M_w=5600\left(\frac{4\ (R^3-r^3)}{3\pi(R^2-r^2)}-\frac{2r}{\pi}\right)=5600\times1.55=$ 8700 foot-pounds. The average moment per foot $=8700/10$ $=870$ foot-pounds. Also $P=500\times2\pi\times5=15{,}700$ pounds, and $M_p=15{,}700\times\frac{4}{\pi}=20{,}000$ foot-pounds, or 2000 foot-pounds per foot on the average.

Expressed in the same terms as the coefficients C of Fig. 11, the average coefficient determined by the above calculation is for load W, $C=\frac{870}{11{,}800}=.074$ and for P, $C=\frac{2000}{15{,}700}=.127.$ Fig. 13 shows the variation in moment normal to AE, as given by the values C_c and C_r in Fig. 11 for both W and P; also the average as calculated herein. This average will be found to agree with the average of the ordinates to the curves.

The results herein given are theoretical results based on assumptions of a homogeneous plate and a certain character

of support. In the actual case the plate is not homogeneous and the supports cannot act exactly as assumed. Concrete is a somewhat plastic material and this tends to equalize the stresses. It would seem that the action of the support is somewhere between that assumed in Case (*A*) and in Case (*B*), so that the radial stresses on the portion *BD* (Fig. 13) will be

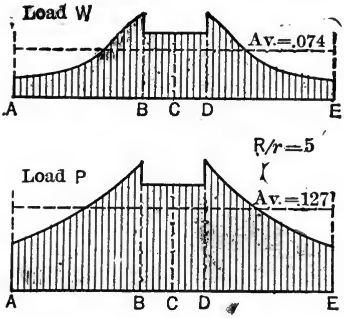


FIG. 13.

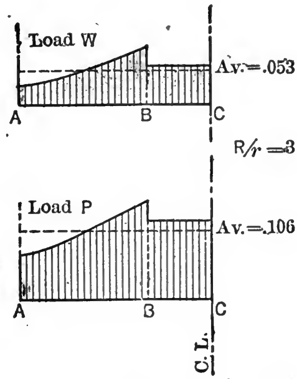


FIG. 14.

larger and the circumferential stresses on *AB* and *DE* somewhat smaller than shown in this figure.

Fig. 14 shows the variation in moment stress in the case of a slab in which $R/r=3$. The relative variation in the value of the stress is considerably less than shown in Fig. 13.

(2) Let it be required to design a square footing to carry a column load of 100 tons on a soil having a safe carrying capacity of $1\frac{1}{2}$ tons per square foot. The column is 2 feet square at its junction with the footing. The allowable stresses are as follows: $f_c=650$, $f_s=16{,}000$, shearing stress as a measure of diagonal tension $=v=90$, using vertical stirrups only, and punching shear $=120$.

Solution.—The area required for the footing $=100/1\frac{1}{2}=66.7$ square feet. Make the footing 8 feet 2 inches square (Fig. 15). The section of maximum moment will be *ABCDEF*. The center of gravity of the area between this section the lower side is found to be 2.15 feet from the center. The upward

pressure on this area is 94,000 pounds. The centroid of the column load, assumed as uniformly distributed on the slab, is .50 feet from the center. The total bending moment on the section is therefore equal to $94,000 \times (2.15 - .50) = 155,000$ foot-pounds. This must be resisted by the reinforcement in the direction perpendicular to AF. The average bending moment per foot is $155,000/8.17 = 19,000$ foot-pounds. The variation of the moment along the line AB must be estimated.

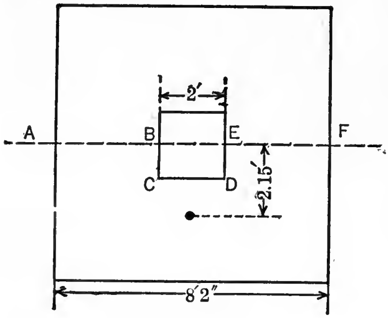


FIG. 15.

In this case the value of R/r is about 4, hence the variation can be roughly estimated from Fig. 11, $R/r=4$. The value of C_c for $x=r$ is about .11 and for $x=4r$ is about .03; average value is about .07. Excess of maximum over average is about 55%. Allowing for a somewhat more even distribution of moment stresses, it will be sufficient to design the section for a maximum moment of, say, 30% in excess of the average, or about 25,000 foot-pounds per foot, and at the edge for a moment of 30% less or 13,000 foot-pounds per foot.

Then with $f_c=650$ and $f_s=16,000$, $R=110$ and $bd^2=\dfrac{25,000\times 12}{110}$ $=227\times 12$. $b=12$ inches. Hence $d^2=227$ and $d=15$ inches. The reinforcement $=0.8\%=1.44$ square inches per foot. Use $\frac{7}{8}$-inch rods spaced 5 inches apart. Near the edge the moment is about one-half as much as at the column, and if the slab is made of uniform thickness the spacing of the steel may be increased accordingly. The same reinforcement will, of course, be required in the transverse direction.

The shearing stresses must now be examined. For shear as a measure of diagonal tension an element of the slab 1 foot wide along CD may be considered. The bending moment at the line CD has been estimated at 25,000 foot-pounds. The

upward force necessary to produce this moment, if uniformly distributed, is equal to 25,000/1.54 = 16,000 pounds, which may be taken as the shear for the present purpose. The shearing stress per square inch $= V/bjd = \dfrac{16{,}000}{12 \times .87 \times 15} = 102$ lbs./in.². This exceeds the allowable value of 90 lbs./in.².

The punching shearing stress around the base of the column is equal to the total upward pressure on the area outside of the column divided by the shearing area of the slab on the square *BCDE*. Upward pressure $= 200{,}000 - 3000 \times 4 = 188{,}000$ pounds. Shearing area $= 8 \times 12 \times 15 = 1440$ square inches. Unit stress $= 188{,}000/1440 = 131$ lbs./in.², which is again in excess of the allowable value.

To secure sufficient shearing strength the thickness of the slab may be increased to 17 inches or the size of the column increased slightly near the bottom. The simplest method is to increase the slab thickness for a short distance near the column. If this is done, the steel reinforcement can be reduced somewhat.

197. Footings Extending under Two or More Columns.—Where a footing slab is continuous under two or more columns

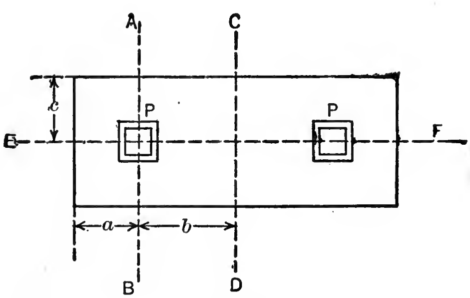


FIG. 16.

the bending moments can be estimated by first calculating the total moments, as for a continuous beam, and then estimating the distribution of these moments across the section in a manner similar to that already illustrated. Consider, for example, a

footing supporting two columns (Fig. 16). The critical bending moments are those on the sections AB, CD, and EF. If each column load $=P$ and the upward pressure per square foot is $p=\dfrac{P}{2c(a+b)}$, then the total moments on the above mentioned lines are approximately as follows:

$$M_{AB}=\frac{P}{a+b}\cdot\frac{a^2}{2};\quad M_{CD}=\frac{P(a+b)}{2}-Pb=\frac{P(a-b)}{2};\quad M_{EF}=Pc.$$

These results are obtained by assuming the column loads concentrated at points. If the area of the column is taken into account the values of M_{AB} and M_{EF} will be somewhat smaller. If it is desired to make the total negative moment on CD equal to the total positive moment on AB, then $\dfrac{a^2}{a+b}=b-a$, or $b=1.41a$.

Having found the total moments, the variation in moment along the several sections may be estimated in the manner illustrated in Art. 196; or use may be made of the rules of practice given in Arts. 203, 204.

Much more complicated arrangements of footings are often found necessary than the simple designs here considered. In all cases, however, the same general method of analysis can be used, namely, the determination of total moments along certain critical lines by ordinary methods of statics or by the adaptation of continuous girder formulas, and then the estimation of the distribution of moments along the sections in question by the methods here described.

D. FLAT SLAB FLOORS.

198. General Description.—The "flat slab floor" is the name given to a type of floor and column design in which the floor is built in the form of a continuous flat slab of uniform, or nearly uniform, thickness, and supported directly upon the columns, the floor and columns being built monolithically. In supporting the load, the floor acts as a continuous plate

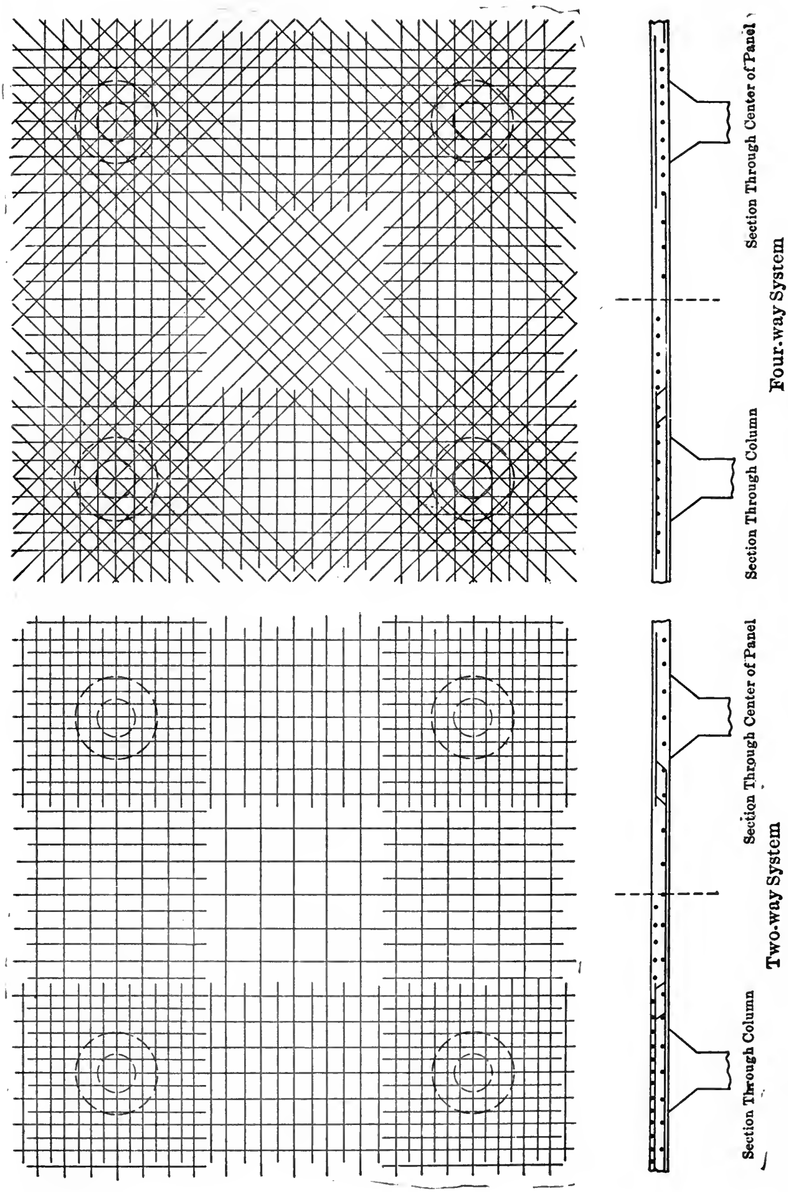


FIG. 17.—Flat Slab Reinforcement.

rather than a combination of beams and slabs, and the reinforcement must be arranged accordingly. Fig. 17 illustrates two common arrangements of columns, slab, and reinforcement. The various arrangements may be divided into two general groups: the two-way system, having rods running in two directions only, and the four-way system, in which diagonal rods are also used. In some systems circular reinforcement is also employed, especially in the areas over the columns.

In general, the bending moment in the slab in the vicinity of the columns will be negative, and in the central portion between columns will be positive. The reinforcing rods will therefore need to be placed near the top surface of the slab in the area over and near the columns, and near the bottom surface in the central areas.

In order to extend the area of support and reduce the stresses in the slab, the columns are usually enlarged at their top, forming column capitals. The stresses in the slab are also sometimes further reduced by thickening a portion of the slab over and near the column, forming what is commonly called a "dropped panel." (See Fig. 24.)

This type of construction has been developed under various patented systems covering arrangement of reinforcement and other details, and the design and construction of buildings of this kind is largely in the hands of specialists connected with construction companies. It is not the purpose of these articles to present details of flat slab design, but only to set forth the general principles underlying the correct analysis of stresses.

199. Nature of Stresses Involved.—Let Fig. 18 represent a portion of a flat-slab floor, including four column supports, assumed as equally spaced. Assume a uniformly distributed load. The column heads underneath the slab are represented by full circles. Considering the nature of the bending moments and deformations at different points, it is plain that if we draw radial lines, AB, AC, AD, from a column center A, the curvature of the slab along such lines will be convex upwards for a certain distance from the center, then concave upwards, then again

convex upwards in the vicinity of the other support. That is to say, there will be points of inflexion along these lines where the radial moment changes sign. If we connect these points of inflexion around the column heads, we will get "lines" of inflexion, as shown by the dotted lines, in Fig. 18, which may be considered roughly as circles. The areas of the slab within these lines of inflexion are stressed in a manner similar to a circular slab supported at the center as described in Art. 194.

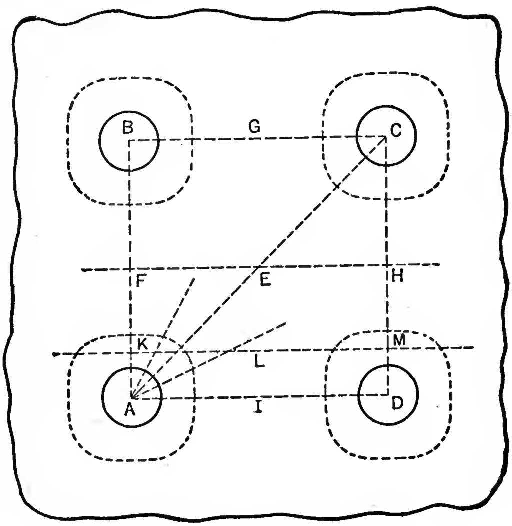


Fig. 18.

The bending moments along all radial lines are negative, and are a maximum at the edge of the column head.

Considering the nature of the bending moments in other than radial directions, we will be aided in our conception if we note that the lowest point of the deflected panel is at the center E, and that the intermediate points, F, G, etc., are higher than E, but lower than the supports. Therefore, if we consider the bending moments along the line FH, we will find negative moments at F and H, and a positive moment at E. Likewise along the line KM there will be negative moments at K and

M and a positive moment at L. Expressed in another way, there will exist ridges along the lines AB, BC, etc., with low points, or "saddles" at the center points. The moments transverse to these ridges are negative at all points. The negative moments transverse to a radial line, as AK, correspond to the "circumferential" moments discussed in Art. 194.

Consider now a single panel, Fig. 19, separated from its supports and the surrounding panels. The total load $= W$. The reactions around the panel consist of shears along the edges of the column heads, EF, GH, etc., and negative bending moments around the entire periphery. Assuming that this panel is surrounded by exactly equal panels, similarly stressed, there will be no shear along the lines FG, HI, etc., and the total shear at each column head will be $W/4$, which may be assumed as uniformly distributed over the quarter perimeter. The negative bending moments around the column heads act in a radial direction, and may be assumed as uniformly distributed. The negative moments along the lines FG, HI, etc., cannot be assumed as uniform; they are evidently a maximum near the columns, and a minimum at the center. The bending moments transverse to a center line MN are all positive and somewhat greater at M and N than at the center.

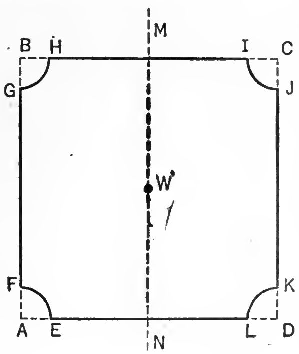


FIG. 19.

An exact calculation of maximum moments is impracticable, as an exact theory of the problem is too complex for solution. It is possible, however, from the principles of statics, to deduce correct values for total bending moments. Having obtained these, we may then estimate the distribution of these moments by a consideration of the nature of the deformations involved, by information obtained from tests, and from the analysis of the circular slab of Art. 196.

200. The Total Bending Moment.—Let Fig. 20 represent

one-half of a panel. In addition to the forces already noted, there will be positive bending moments along the line IJ, but no shear (by reason of symmetry). The half panel load will act at its center of gravity O, and the shear reactions R, along the column head, may be considered as acting at their centroids, a distance b from the center line AB. Assuming a uniform shear, it can be shown that $b = c/\pi$, where c = diameter of column capital.

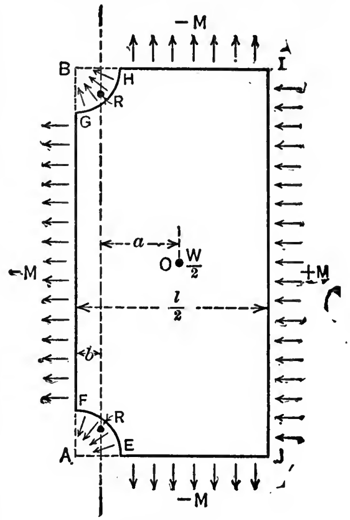


FIG. 20.

Considering the moments transverse to the axis AB, the external moment is $Wa/2$. This is resisted by the positive moments along IJ and the negative moments along $HGFE$. The negative moments along HI and EJ have no components transverse to AB. The radial moments along HG and FE may be resolved into components transverse and parallel to AB. Assuming the radial moments as uniform, it will be found that the transverse components will also be uniform in amount per foot of projection on the axis AB. We may therefore say that the sum of the positive moments along IJ and the negative moments along AB (including the resolved radial moments around the column capital) is equal to the external moment $Wa/2$. Taking account of the portion of the load supported directly by the column, and considering the usual size of column capital, the value of the total moment $Wa/2$ is given very closely by the equation:

$$M = \tfrac{1}{8}wl(l - \tfrac{2}{3}c)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where M = numerical sum of negative and positive moments;
w = load per unit area, assumed as uniform;
l = distance center to center of columns;
c = diameter of column capital.

201. *Oblong Panels.*—The above analysis holds true for oblong panels as well as square panels, providing there are a number of consecutive panels of the same size, so that the shears along the center lines may be taken at zero. Referring to Fig. 21, the equations for the sums of the negative and positive moments in the two directions are:

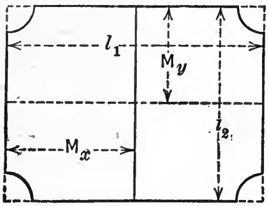


FIG. 21

$$\left.\begin{aligned} M_x &= \tfrac{1}{8}wl_2(l_1-\tfrac{2}{3}c)^2 \\ M_y &= \tfrac{1}{8}wl_1(l_2-\tfrac{2}{3}c)^2 \end{aligned}\right\}, \quad \ldots \ldots \quad (2)$$

where l_1 and l_2 are the dimensions of the panel center to center and M_x and M_y the sums of the moments on sections parallel to l_2 and l_1 respectively.

202. Distribution of the Bending Moment.—The analysis, up to this point, has been based on statics alone, and gives correct values for total bending moments under the conditions assumed. The distribution of these moments cannot be so readily determined; it depends upon the relative rigidity of the different parts of the structure. There is to be considered, first, the question of the relative amounts of positive and negative moments and, second, the variation of these moments along the slab.

203. *Relative Amounts of Positive and Negative Moments.*—The relative proportions of the bending moment carried along the line *EFGH* (negative) and the line *IJ* (positive) can be estimated by comparing the slab to a beam continuous over a number of supports, or a beam fixed at the ends. In the latter case, the negative moment is two-thirds of the total, and the positive one-third. Considering the relative flexibility of the portion of the slab from *F* to *G*, it is reasonable to assume a distribution of bending moment of five-eighths of the total as negative and three-eighths as positive. The total negative and positive moments would then be, for a square panel,

$$\left.\begin{aligned} M_- &= \tfrac{5}{64}wl(l-\tfrac{2}{3}c)^2 \\ M_+ &= \tfrac{3}{64}wl(l-\tfrac{2}{3}c)^2 \end{aligned}\right\}, \quad \ldots \ldots \quad (3)$$

The foregoing analysis for total moment is theoretically correct, and gives a value for the total moment which must be resisted in some manner by the construction. There are, however, certain practical reasons why the coefficients here obtained may be safely reduced if the usual working stresses are employed; or the working stresses may be increased if the full values for the moments are used. In this type of construction, the effect of local variations in the concrete is less than in the case of ordinary beams, the tensile stresses in the concrete are of much greater assistance, and arch action of slabs tends to reduce the steel stresses. Experience with construction of this type, and observations from tests indicate that the theoretical values may safely be reduced 15% as recommended by the Joint Committee giving for the working values of the negative and positive moments

$$\left.\begin{array}{l} M_{-}=\frac{1}{15}wl(l-\frac{2}{3}c)^2 \\ M_{+}=\frac{1}{25}wl(l-\frac{2}{3}c)^2 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

204. *Distribution of Moment Along the Slab*—Having determined the total negative and positive moments, it remains to estimate their distribution along the slab. For this purpose, it will be helpful to consider the slab in two parts (Fig. 22): a strip a, to include the column head, and a central strip b. The strip a is evidently more rigid than b, and the proportion of the moments carried by a will therefore be greater than that carried by b. If the width of each strip is taken as $l/2$, then it is evident that more than one-half the negative moment on $HGFE$ will be carried by the two quarters, HGL and KFE, and less than half by the center half, LK. Likewise more than one-half the positive moment on IJ will be carried by the quarters IM and NJ, and less than half by MN. Just what distribution should be made between the sections adjoining the columns and the central section is a matter of judgment, and a considerable variation in proportion can be assumed without serious consequences. A slightly deficient reinforcement at

one point, or a slight overstressing of the concrete, allows such portion to deform more readily and thus transfers more moment to the other adjoining and more rigid portions. The important requirement is that the total moment be adequately taken care of. In practice, from 60% to 80% of the negative moments are taken care of in the outer quarters, and 20% to 50% in the central section. The positive moments may be assumed

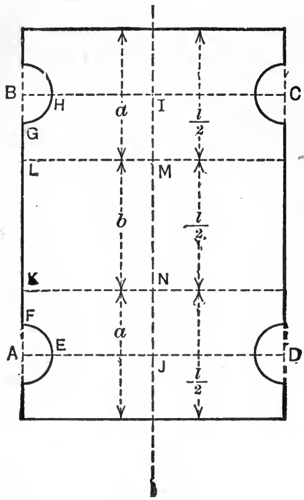


Fig. 22.

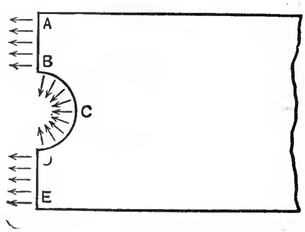


Fig. 23.

more uniformly distributed, from 35% to 50% in the inner section and 50% to 65% in the outer sections. (See report of Joint Committee in Appendix.)

Considering in more detail the bending moments around and near the column capital (Fig. 23), it may be assumed that the radial moments on the line *BCD* are approximately uniform. The moments on *AB* and *DE* follow a different law; they correspond to the circumferential moments of the circular slab of Art. 194. In that article it was shown that the circumferential moments are probably less in intensity than the radial moments immediately at the edge of the column head. The latter, however, decrease rapidly in amount with increased distance from the center; the former change more gradually. Observations on buildings show that, under overload, circular cracks above the column capital form first, but that cracks along the lines *BA* and *DE* form soon afterward, showing circumferential stresses approaching in magnitude the radial stresses.

It may be concluded, therefore, that if the strip AE is not taken too wide in comparison to the diameter of the column capital, the components of the moments normal to the line AE may be taken as uniformly distributed. Between the points L and K, Fig. 22, the moment is more nearly uniform, and may be assumed as such. The positive moments on IJ will obviously be more nearly uniform than the negative, and may fairly be assumed as uniform in each strip.

205. Calculation of Reinforcement.—In calculating stresses in the reinforcement, all bars crossing the sections in question may be counted, provided adequate provision is made for bond strength. Rods crossing diagonally may be considered to have a value determined by multiplying their area by the sine of the angle which they make with the side of the panel considered. Thus, a diagonal bar on line AC (at 45°), Fig. 22, would be counted for negative moment on line KA and positive moment on line MN at .707 of its full area.

206. *Bond Strength of Bars.*—To provide adequate bond strength where bars are discontinued, they should extend somewhat beyond the line of inflection and sufficiently beyond the point of maximum moment to develop their full strength. Bars should not be bent up all at one place, but bends should be staggered.

207. Shear and Diagonal Tension.—The slabs must be designed to resist diagonal tension in proportion to the bending moments assumed. The shear will be a maximum in the column-head section. Taking the total negative moment at $\frac{1}{15}wl(l-\frac{2}{3}c)^2$, and assuming as much as 75% to be carried by the column-head section, we get a negative moment of $\frac{1}{20}wl(l-\frac{2}{3}c)^2$ for this half portion. For $c=\frac{1}{5}l$, this would equal about $\frac{1}{27}wl^3$. The negative bending moment in a slab of dimensions $l\times l/2$, fixed at the ends, is $\frac{1}{24}wl^3$, or slightly more than the above figure. It may then be concluded that the diagonal tension around the column head is not greater than that in a slab of width $l/2$, fixed at the ends and loaded with a load w per unit area. The end shear in the latter case is $.25wl^2$,

and the shear per unit width is $.5wl$. We may therefore say that the shear per unit width, as a measure of diagonal tension, may be taken at $.5wl$ for square panels. For dropped panels, where the moment taken by the column head section is larger, the shear per unit width will be larger, the value of $.6wl$ per unit width being recommended by the Joint Committee. For oblong panels the shear per unit of width will be $.5wl_2$, where l_2 is the dimension perpendicular to the section in question.

For the "punching" shear, or true shearing stresses around the column, the shear per unit width will be obtained by dividing the total panel load by the circumference of the column capital.

208. The Drop Panel.—Where the drop panel is used, as described in Art. 198, the stresses are reduced in the vicinity of the column. The column-head section will then have a cross-section as shown in Fig. 24. The effect of the drop panel is to stiffen considerably the column-head section of the slab and therefore to cause a somewhat greater proportion of the bending moment to be carried by this part and less by the central section. Recommended proportions are given in the report of the Joint Committee.

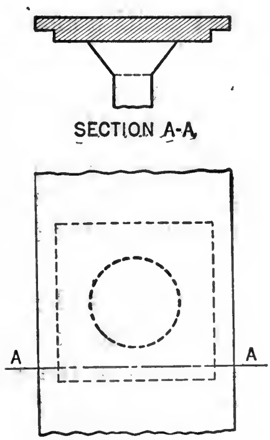


Fig. 24.

Having determined upon the amount of the bending moment, the stresses in the section at the column are determined by considering the entire section as acting as a unit. At the edge of the drop is another point of maximum stress, and the stresses here in the slab should be determined. A drop panel of small area or relatively great depth may result in bringing the critical stresses in the slab at the edge of the drop.

209. Rules of Practice in Design.—The rational approximate analysis of the flat slab as herein given was not developed until this form of construction had long been in use. Designs have

been largely in the hands of construction companies, and have been based generally on empirical rules checked to a greater or less extent by tests on completed structures and test panels. Gradually these rules of practice have become better systematized and placed on a rational basis, and have been incorporated into building codes of cities, thus enabling various designs to be checked and compared. One of the latest and best codes of rules is that of the City of Chicago, amended January 1, 1918. The regulations adopted by the American Concrete Institute, 1917, are also rational in form and cover the subject comprehensively. Both of these sets of rules are less conservative than those of the Joint Committee, but they represent good current practice, and in the opinion of most engineers engaged in this work are adequate to secure safe and satisfactory structures. The principal requirements of these two sets of rules and those of the Joint Committee are herewith given for square panels with round columns:

Nomenclature:

w = total load per square foot of floor;
l = span length c. to c. of columns, in feet;
c = diameter of column capital where it is $1\frac{1}{2}$ inches deep;
t = total thickness of slab in inches.

The designation of the different sections is shown in Fig. 25.

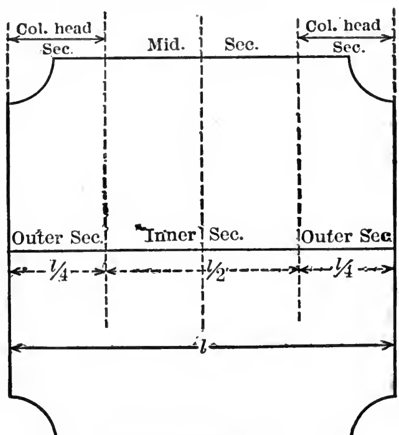


Fig. 25.

Joint Committee.

Minimum Thickness.—For slab without dropped panels,

$$\text{Minimum } t = .024l\sqrt{w} + 1\tfrac{1}{2}.$$

For slab with dropped panels,

$$\text{Minimum } t = .02l\sqrt{w} + 1.$$

For the dropped panel of width equal tc $.4l$,

$$\text{Minimum } t = .03l\sqrt{w} + 1\tfrac{1}{2}.$$

In no case may floor slabs be less than 6 inches in thickness nor less than $\frac{1}{32}l$. Roof slabs may not be less than $\frac{1}{40}l$.

Bending Moments —Total positive moment $= \frac{1}{25}wl(l - \frac{2}{3}c)^2$.

On inner section at least 25%.

On outer section at least 55%, and with dropped panel at least 60%.

Total negative moment $= \frac{1}{15}wl(l - \frac{2}{3}c)^2$.

On mid-section at least 20%.

On column-head section at least 65%, and with dropped panels at least 80%.

Shear.—For calculating shear as a measure of diagonal tension, the shear per foot of width is $V = .5wl$.

For dropped panels $V = .6wl$.

Then $v = \dfrac{V}{bjd}$, as in ordinary beams. For punching shear around the periphery of the column capital, use total vertical shear increased by 25%.

Working Stresses.—The same as for other concrete construction.

Fibre stress in slabs, 32.5% of compressive strength. Adjacent to supports, increase by 15%.

Shear for diagonal tension, without web reinforcement, 2% of compressive strength.

Punching shear, 6% of compressive strength.

Bond.—For plain bars, 4% of compressive strength; for deformed bars, 5% of compressive strength.

Chicago Ruling. (1918)

Minimum Thickness.—Minimum $t = \frac{1}{44}l\sqrt{w}$, but in no case less than 6 inches or $\frac{1}{32}l$ for floor slabs and $\frac{1}{40}l$ for roof slabs.

Bending Moments.—For panels with standard drops:

Total positive moment, two-way system $= \frac{1}{40}wl^3$.

On inner section $M = \frac{1}{120}wl^3$;

On outer section $M = \frac{1}{60}wl^3$.

Total positive moment, four-way system $=\frac{1}{48}wl^3$.

On inner section $M=\frac{1}{120}wl^3$;

On outer section $M=\frac{1}{80}wl^3$.

Total negative moment, two-way and four-way systems $=\frac{1}{24}wl^3$.

On mid-section $M=\frac{1}{120}wl^3$;

On column-head section $M=\frac{1}{30}wl^3$.

Working Stresses.—Fiber stresses in slabs, 35% of compressive strength.

Punching shear, 6% of compressive strength.

Bond, 70 lbs./in.^{2} for ordinary structural steel.

American Concrete Institute. (1917)

Minimum Thickness.—Minimum $t=.02l\sqrt{w}+l$, but not less than $\frac{1}{32}l$ for floor slabs and $\frac{1}{40}l$ for roof slabs.

Bending Moments.—Sum of positive and negative moments $=.09wl(l-\frac{2}{3}c)^2$.

Positive Moments.—On inner section not less than 12% of total.

On outer section not less than 18% of total.

Negative Moments.—On mid-section not less than 10% of total.

On column-head section not less than 40% of total for slabs, and 50% of total where drop is used.

Shear.—Same as Joint Committee requirements.

Working Stresses.—Fiberstress in slabs, $37\frac{1}{2}$% of compressivestrength, increased by 10% in column-head section.

Shear for diagonal tension, without web reinforcement, 2% of compressive strength.

Punching shear, $7\frac{1}{2}$% of compressive strength.

Bond, for plain bars, 4% of compressive strength; for deformed bars, 5% of compressive strength.

Comparing the foregoing rules, it will be found that the Joint Committee's recommendations are considerably more conservative than either of the others. If $c=.225l$, a common value, then we find the following values for bending moments:

Joint Committee $M_+ = .0288wl^3$;

$M_- = .048wl^3$;

Total $M = .0770wl^3$.

Chicago Ruling

Two-way system:

$$M_{+} = .0250wl^3;$$

$$M_{-} = .417wl^3;$$

$$\text{Total} \quad M = .0667wl^3.$$

Four-way system:

$$M_{+} = .0208wl^3;$$

$$M_{-} = .0417wl^3$$

$$\text{Total} \quad M = .0625wl^3.$$

American Concrete Institute

$$\text{Total} \quad M = .0648wl^3.$$

The unit stresses allowed in bending are as follows:

Joint Committee, 32.5% of compressive strength;

Chicago Ruling, 35% of compressive strength;

American Concrete Institute, 37.5% of compressive strength.

As regards bending moment, it is seen that the Chicago rule provides for a total of 13% to 19% less, and the American Concrete Institute 16% less than the Joint Committee, and the unit stresses in the concrete are considerably higher.

210. Tests of Flat Slabs.—On account of the difficulty of making theoretical analyses, experimental results have been more relied upon in the design of flat slabs than in most types of construction. It has been customary to impose an acceptance test of the completed work in which test loads of $1\frac{1}{2}$ to 2 times the working load have been used and the deflections measured. Such tests are, however, of little value in design beyond serving as a check on very inadequate proportions, as the manner of distribution of stress is not brought out. Besides these loading tests, a considerable number of tests have been made in which strain gages have been used and the loading carried considerably farther. The general results of most of these tests show that with a test load of 2 to $2\frac{1}{2}$ times the de-

sign live load, the measured stresses in the reinforcement are much below the theoretical values. This is due to several causes, principally the resistance due to tension in the concrete, the support of adjacent unloaded panels, and the fact that strain measurements cannot give maximum stresses at cracks but average stresses over a considerable length. The same general results but less marked in degree are always obtained in similar tests on rectangular beams until loads are reached approaching the ultimate strength. To secure results which can be expected to check with theory it is necessary to carry the tests to a point where the concrete is thoroughly cracked. Such a test is that made by Professor A. N. Talbot * in which the test was carried practically to destruction and great care taken in the strain measurements. In this case four panels were loaded with a load equal to about 3½ times the design live load, producing deflections of about 1 inch at centers of panels and cracking the floors at both top and bottom. At the maximum load the total resisting moment as measured by the strains in the steel were about 98% of those as calculated by the Joint Committee rules, the negative moment being 13% higher, and the positive moment 27% lower. The maximum stresses in bars were 15% to 25% greater than the average. Stresses in the concrete reached ultimate values adjacent to the columns and were about 80% to 90% as great at the top of the slab midway between columns. Considering the nature of such tests and the fact that any effect of tensile stress in the concrete or the supporting effect of adjacent panels acts always to decrease the resisting moment as calculated from the tests, these results are as close to the theoretical (which is about 17% higher than the rules of the Joint Committee) as could be expected,

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* Bul. No. 106, Eng. Exp. Sta., Univ. of Ill., 1918.

References to other valuable tests are as follows:

Bul. No. 84, Eng. Exp. Sta., Univ. of Ill., 1916.
A. R. Lord, Am. Concrete Inst., 1917, p. 45.
W. K. Hatt, Am. Concrete Inst., 1918, p. 164.

and confirm the general correctness of the method of analysis as herein given.

While this method of analysis is undoubtedly substantially correct, the results of these tests, as well as others, show that for loads considerably beyond the usual working loads the stresses in the steel are far below the calculated stresses, and furthermore that this type of structure is exceedingly reliable and capable of heavy overload without danger of sudden failure. The shearing stresses are generally low, and the danger of shear failures which exist in ordinary beam construction hardly exists. It is, therefore, proper and admissible that higher unit stresses should be used in this type of design, or that some reduction be made in the theoretical moment coefficients. The latter is the usual method of modification.

211. Examples.—(1) Let it be required to design a flat slab floor on the four-way system to meet the following requirements:

Panel, 20 feet by 20 feet.

Diameter of column capital $=.225l=4.5$ feet;

Live load $=150$ lbs./ft.²;

$f_s=16{,}000$ lbs./in.²;

$f_c=750$ lbs./in.².

Assume a concrete of 2300 lbs./in.² compression strength.

Coefficients for bending moment and general details as recommended by the Joint Committee. (See Appendix.)

Solution.—A slab of uniform thickness will be used and the general arrangement of rods as shown in Fig. 17, "four-way system." The thickness of the slab will be determined by the negative moment in strip A (Fig. 26), which is the column-head section, but it must not be less than given by the requirement $t=.024l\sqrt{w}+1\frac{1}{2}$. A thickness of 9 inches will be assumed as a first trial. Then dead load $=\frac{9}{12}\times150=112$ lbs./ft.² and $w=262$ lbs./ft.². By Eq. (4), Art. 203, $M_-=\frac{1}{15}wl(l-\frac{2}{3}c)^2$ $=\frac{1}{15}\times262\times20(20-3)^2=101{,}000$ ft.-lbs. This will be divided in the proportion of 75% in strip A and 25% in strip B. This gives for strip A, $M_-=.75\times101{,}000=76{,}000$ ft.-lbs. From Plate II, the value of R for $f_s=16{,}000$ and $f_c=750$ is 134; also $p=.097$ and $j=.86$. Then $bd^2=76{,}000\times\frac{12}{134}=6800$ and $d^2=\frac{6800}{120}=56.7$. Therefore $d=7\frac{1}{2}$ inches. Over the columns there will be four layers of rods, so that the average depth of steel from the surface may be taken at about 2 inches. This gives a total thickness of slab of $9\frac{1}{2}$ inches. The corrected value of the

dead load will be 120 and the total load 270 lbs./ft.[2] The formula for minimum thickness gives $t = .024 \times 20\sqrt{270} + 1\frac{1}{2} = 9.4$ inches. The value

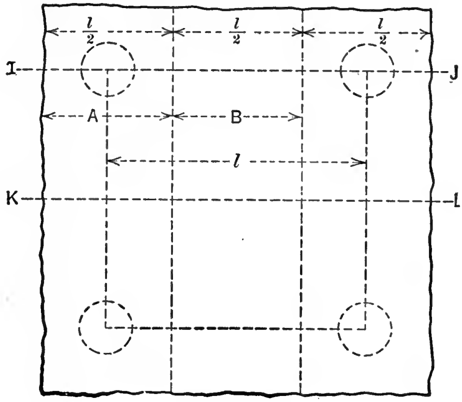


FIG. 26.

of $9\frac{1}{2}$ inches will then be used. The negative and positive moments will then be as follows:

Negative Moments.—Total $= \frac{1}{15} \times 270 \times 20(20-3)^2 = 104{,}000$ ft.-lbs.

Strip A, 75% of 104,000 = 78,000 ft.-lbs.

Strip B, 25% of 104,000 = 26,000 ft.-lbs.

Positive Moments.—Total $= \frac{1}{25} \times 270 \times 20(20-3)^2 = 62{,}500$ ft.-lbs.

Strip A, 55% of 62,500 = 34,500 ft.-lbs.

Strip B, 45% of 62,500 = 28,000 ft.-lbs.

The required steel areas are as follows, using the formula $A_s = \frac{M}{jdf_s}$:

Strip A, section IJ:

$$A_s = \frac{78{,}000 \times 12}{.86 \times 7.5 \times 16{,}000} = 9.05 \text{ sq. in.}$$

Strip A, section KL:

$$A_s = \frac{34{,}500 \times 12}{.90 \times 8.0 \times 16{,}000} = 3.60 \text{ sq. in.}$$

Strip B, section IJ:

$$A_s = \frac{26{,}000 \times 12}{.90 \times 8.0 \times 16{,}000} = 2.70 \text{ sq. in.}$$

Strip B, Section KL:

$$A_s=\frac{28{,}000\times 12}{.90\times 8.0\times 16{,}000}=2.92 \text{ sq. in.}$$

For all sections excepting over the column the value of j has been taken at .90, and the effective depth at 8 inches, as the percentage of steel required in these sections is low. Over the column where there are four layers of rods, the depth has been taken at 7.5 inches, and j at .86.

The number of diagonal rods is determined by the positive moment in strip B which requires 2.92 square inches. Using $\frac{1}{2}$-inch round rods, the number required in each band $=\frac{2.92}{2\times .196\times \sin 45°}=10.6$. Use 11 rods in each diagonal band.

Strip A, section IJ, requires 9.05 square inches. If all the diagonal rods are bent up over the column, the area furnished by the 11 diagonal rods is $11\times 2\times .707\times .196=3.03$ square inches, leaving $9.05-3.03=6.02$ square inches to be furnished by the direct bands. Trying $\frac{5}{8}$-inch rods, the number required $=\frac{6.02}{.307}=20$. On section KL the number required $=\frac{3.60}{.307}=12$. The direct bands may then be made up of 12 rods, bending up all but two, and overlapping them at the columns so as to give the necessary 20 at section IJ. If the direct band of 12 rods is made continuous over two panels, a common practice, then the entire 12 may be bent up and short rods added over the column. Various other arrangements may be used as is found convenient. The bending points of the rods should be staggered somewhat on both sides of the line of inflection. For strip B, section IJ, the number of $\frac{1}{2}$-inch rods required $=\frac{2.70}{.196}=14$. Short rods will be used.

The slab must now be tested for shear. As a measure of diagonal tension the value of the shear is $v=\frac{.25W}{bjd}$, where $W=$ total load on panel, $=270\times 400-270\times \pi\times 2.25^2=104{,}000$ lbs. $v=\frac{.25\times 104{,}000}{120\times .86\times 7.5}=33.7$ lbs./in². This requires no shear reinforcement. For punching shear the shearing stress around the periphery of the column $=\frac{104{,}000\times 1.25}{\pi\times 4.5\times 12\times 9.5}$ $=81$ lbs./in.², also a safe value.

(2) With the same live load, size of panel, and unit stresses as in (1), use a "dropped panel" proportioned in accordance with the rules of the Joint Committee.

The width of dropped head will be $.4\times 20=8$ ft. Assuming the average dead load to be 100 lbs./ft.², $w=250$ lbs./ft.² The minimum thickness of slab $=.02l\sqrt{w}+1=.02\times 20\times 15.8+1=7.3$ inches, and the thickness of the drop $=.03l\sqrt{w}+1\frac{1}{2}=11.0$ inches. This gives an average thickness of about 7.9 inches and a weight of practically 100 lbs./ft.² as assumed. The thickness will then be taken for trial at $7\frac{1}{2}$ inches for the slab and 11.0 inches for the drop.

The total negative moment $=M_{-}=\frac{1}{15}\times 250\times 20(20\times 3)^2=97{,}000$ ft.-lbs. Assume 80% carried by the column head section $=77{,}600$ ft.-lbs. For carrying compressive stresses the effective width of the column head section is practically 8 feet, the width of the drop. Then, as in Ex. (1), $bd^2=77{,}600\times\frac{12}{134}=7000$; $d^2=\frac{7000}{8\times 12}=73.0$ and $d=8.5$. The thickness of 11.0 inches is therefore ample as regards concrete stress.

The various moments will be as follows:

Negative Moments.—Total $=\frac{1}{15}\times 250\times 20(20-3)^2=97{,}000$ ft.-lbs.

Strip A, 80% of $97{,}000=77{,}600$ ft.-lbs.

Strip B, 20% of $97{,}000=19{,}400$ ft.-lbs.

Positive Moments.—Total $=\frac{1}{25}\times 250\times 20(20-3)^2=58{,}000$ ft.-lbs.

Strip A, 60% of $58{,}000=34{,}800$ ft.-lbs.

Strip B, 40% of $58{,}000=23{,}200$ ft.-lbs.

The required steel areas are:

Strip A, section IJ:

$$A_s=\frac{77{,}600\times 12}{.86\times 9\times 16{,}000}=7.5 \text{ sq. in.}$$

Strip A, section KL:

$$A_s=\frac{34{,}800\times 12}{.90\times 6\times 16{,}000}=4.85 \text{ sq. in.}$$

Strip B, Section IJ:

$$A_s=\frac{19{,}400\times 12}{.90\times 6\times 16{,}000}=2.7 \text{ sq. in.}$$

Strip B, section KL:

$$A_s=\frac{23{,}200\times 12}{.90\times 6\times 16{,}000}=3.22 \text{ sq. in.}$$

The required area for each diagonal band $=\frac{3.22}{1.41}=2.28$ square inches. Use twelve $\frac{1}{2}$-inch round rods $=2.35$ square inches, bending same up

over the column. In strip A, section IJ, the total required area is 7.5 square inches, of which the diagonal bands furnish $2.35\times1.41=3.3$ square inches, leaving $7.5-3.3=4.2$ square inches for the direct bands. This is furnished by 22 $\frac{1}{2}$-inch round rods. The strip A, section KL, requires $\frac{4.85}{.196}=25$ $\frac{1}{2}$-inch rods in the direct band. All but three of these may then be bent up to furnish the negative reinforcement over the column. The strip B, section IJ, requires $\frac{2.7}{.196}=14$ $\frac{1}{2}$-inch short rods placed in the upper part of the beam. The length of these rods must be 4 ft. 10 in. each side of the column center line (equal to $\frac{1}{5}$ panel length plus 20 diameters of rod). An examination of shearing stresses will show that these are within safe limits.

Comparing this design with that of Ex. (1) it will be found that the amount of steel required is practically the same, but is more uniformly distributed in (2) than in (1). The amount of concrete is considerably less, averaging 8 inches in thickness as compared to $9\frac{1}{2}$ inches.

CHAPTER IX.

BUILDING CONSTRUCTION.

212. The foregoing chapters have dealt with the analysis of various elements of construction without much reference to the structure as a whole. The use of these elements in many forms of construction is simple and direct and requires no further discussion; but in other cases, when these elements are combined into whole structures, certain problems of stress determination arise which differ somewhat from those involved in similar structures of steel or timber. In the present chapter the use of reinforced concrete will be considered with especial reference to building construction. The various features may be grouped under the following heads: (1) Beams forming a continuous surface, as floor- and roof-slabs; (2) Floor-beams and girders; (3) Columns; (4) Footings; (5) Walls and partitions. In the discussion of these various elements consideration will be given to the determination of stresses, the design of members, and the arrangement of connective details.

213. **General Arrangement of Concrete Floors.**—Concrete floors may for present purposes be divided into three types: (1) Floors consisting of concrete slabs supported on steel beams, (2) concrete slabs built monolithically in combination with a concrete framework of beams and girders, and (3) the flat-slab floor where slabs are supported directly by concrete columns. In the first case the building is of the ordinary steel frame construction, consisting of columns, girders and cross-beams, concrete being used between beams for the continuous floor and generally at the same time as fire-proofing material. The beams are generally spaced about 6 feet apart. Various

designs are illustrated in Art. 227. If a reinforced concrete framework is used, the same system of girders and beams is generally employed, the beams being spaced 4 to 8 feet apart (Fig. 1*a*). In some cases it is found desirable to use cross-beams at columns only, thus forming with the girders square or nearly square panels with slabs supported on four sides (Fig. 1*b*); and again, where the columns may be spaced close together, and where no lateral force is to be resisted, as in a covered reser-

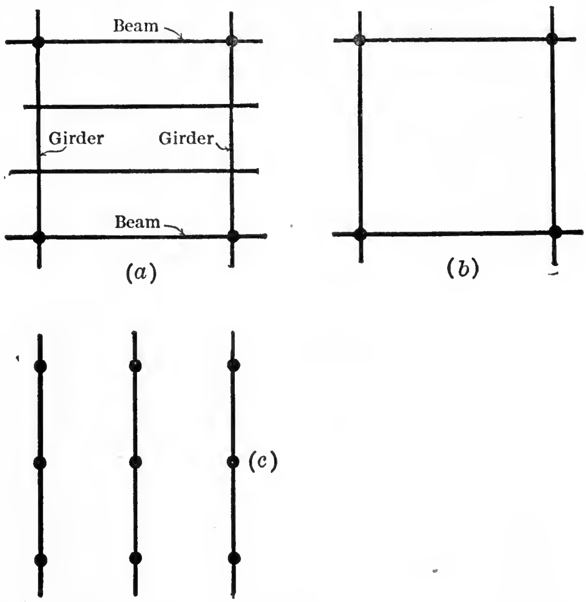


FIG. 1

voir, it may be economical to run beams in one direction only (Fig. 1*c*). In (*a*) the slabs are supported mainly by cross-beams and are generally so designed, a small amount of reinforcement to prevent cracks being placed transverse to the girders. Generally an odd number of panels is advantageous, as this avoids a concentration at the middle of the girder, and reduces the bending moment therein. In the flat-slab system no beams are used except around openings and, frequently,

in the outside panels. This system is fully discussed in Chapter VIII.

In all the types of beam and slab design the slabs and beams are built monolithically and therefore are continuous from span to span, and should be designed as continuous girders. If designed as simple beams, as is generally done with a steel frame, and no provision made for negative moments at supports, cracks will form in the concrete at these points, which tend to bring about disintegration of the concrete and which weaken the beams in shear. The best practice is, therefore, to treat the slabs and beams as monolithic continuous structures, and to design them accordingly. An important problem in the design of slabs and beams is, therefore, the determination of maximum bending moments in the series of panels or spans considered as continuous beams.

ANALYSIS OF CONTINUOUS BEAMS.

214. General Conditions.—The exact determination of the maximum bending moments at all sections of a beam continuous over several spans is a tedious and time-consuming problem. For several reasons such a complete solution is generally unnecessary and of little value. In the ordinary case the beams in question (especially floor slabs), are continuous over several spans, and the loading required to produce the theoretical maximum stresses involves unreasonable assumptions as to position of live load. It would be necessary, in general, not only to load alternate panels completely, leaving intermediate panels entirely unloaded, but it would be necessary also to have such load conditions extend over a very considerable width transversely of the beams in question, as the monolithic character of a concrete floor produces a wide lateral distribution of concentrated loads. In most cases, therefore, a sufficiently exact analysis may be arrived at by considering certain simple cases of continuous beams. Where span lengths vary considerably, and in other special cases, a more exact analysis should be made.

215. General Formulas.—The calculation of moments, shears, and reactions for continuous beams is based on the theorem of three moments, which expresses the relation between the bending moments at any three consecutive supports and the loads on the two included spans. Referring to Fig. 2, let supports 1, 2, and 3 be any three consecutive supports of a continuous girder of any number of spans; l_1 and l_2 the included span lengths, and M_1, M_2, and M_3 the bending moments at the respective supports. In (*a*) the load consists of uniform loads of w_1 and w_2 per unit length on the two spans, and in (*b*) the load consists of any number of concentrated loads represented by P_1 and P_2, respectively,

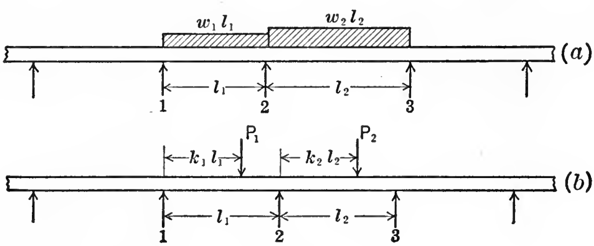


FIG. 2.

and distant kl_1 and k_2l_2, respectively, from the supports on the left. The theorem of three moments for these two cases is

For uniform loads:

$$M_1l_1+2M_2(l_1+l_2)+M_3l_2=-\tfrac{1}{4}w_1l_1^3-\tfrac{1}{4}w_2l_2^3. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1a)$$

For concentrated loads:

$$\begin{aligned} M_1l_1+2M_2(l_1+l_2)+M_3l_3=&-\Sigma P_1l_1^2(k_1-k^3) \\ &-\Sigma P_2l_2^2(2k_2-3k_2^2+k_2^3) \qquad (1b) \end{aligned}$$

By the use of these equations the bending moments at all supports can be determined, and thence the shears and reactions. For example, a two-span girder supported at the ends is solved at once by applying the equation to the moments at the three supports, the values of M_1 and M_3 being zero. In a three-span girder the theorem is applied to the first and second spans and to the second and third spans, placing M_1 and M_4 = zero.

Two equations are thus formed with two unknown moments M_2 and M_3, and these moments determined. In a similar manner three equations are written for a four-span girder, etc. If the ends are fixed so that the end moments are not zero, two additional equations are obtained by assuming two additional end spans, each of zero length, and then considering the structure as supported at the ends, making the end moments zero.

Having determined the moments at supports, the moments at other points and the shears may be determined as follows:

In Fig. 3 the end moments M_1 and M_2 are known. They are represented as negative moments, the usual case. The beam is also loaded in any manner. It is required to determine the shears V_1 and V_2 and the moment M_x at any point C. Let V'_1, V'_2, and M'_x represent the shears and moment due to the vertical

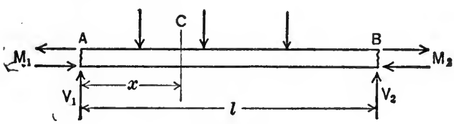


FIG. 3.

loads alone, considering the beam as simply supported. The effect of M_1 and M_2 can then be calculated separately and added. We thus find readily, taking moments about B

$$V_1 = V'_1 + \frac{M_1 - M_2}{l}; \quad \ldots \ldots \quad (2)$$

$$V_2 = V'_2 + \frac{M_2 - M_1}{l}; \quad \ldots \ldots \quad (3)$$

$$M_x = V_1 x - M_1$$

$$= V'_1 x + (M_1 - M_2)\frac{x}{l} - M_1$$

$$= M' - \left[M_1 + (M_2 - M_1)\frac{x}{l}\right]. \quad \ldots \quad (4)$$

For the center point

$$M_c = M'_c - \frac{M_1 + M_2}{2}. \quad \ldots \ldots \ldots \quad (5)$$

Fig. 4 illustrates the case for a uniform load. M_1 and M_2 are the end moments, and the curve DEF is a parabola plotted from the axis DE, with center ordinate $= \frac{1}{8}wl^2$. The ordinates from the axis DE to the curve represent values of M', the moment in a simple beam. The resultant moment, represented by the

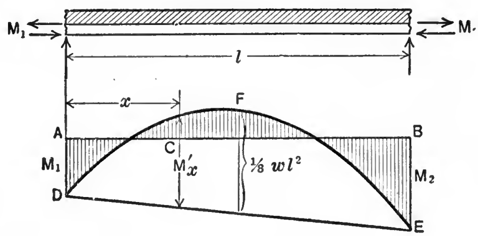


FIG. 4.

ordinates from AB to the curve, is equal to M' minus the ordinate from AB to DE, which is equal to $M_1 + (M_2 - M_1)\frac{x}{l}$, as given in eq. (4). At the center this is $\frac{M_1 + M_2}{2}$ as in eq. (5). For concentrated loads the moment diagram for M' will be plotted from the axis DE in the same manner, the total or resultant moment being represented by the ordinates from AB.

By the application of the theorem for a single concentrated load influence lines may be drawn for moment at any point and from these the exact position of a uniform load or a series of concentrated moving loads, determined for a maximum effect. Ordinarily, this process is not necessary, a sufficiently exact solution being obtained by the methods described herein.

The character of the influence lines for moment and shear is well shown in Fig. 5 in which a six-span girder is considered. Figs. (b) and (c) are the influence lines for moment at (a) and

(*b*), the centers of the first and third spans, Figs. (*d*) and (*e*) for moment at supports 2 and 4 and Figs. (*f*) and (*g*) for shears at (*a*) and (*b*). A maximum moment at the center of a span

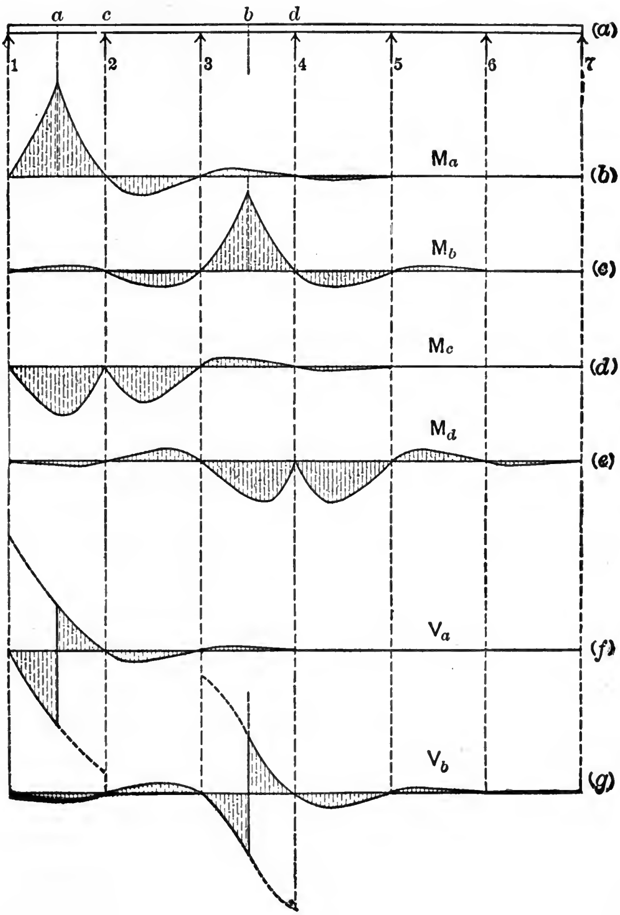


FIG. 5.

requires each alternate span to be loaded, and a maximum moment at the support requires the two adjacent spans to be loaded and then each alternate span. The small effect of loads on remote spans is to be noted. For maximum shears the

general rule of loads in alternate spans is seen to hold true as for moments, but the effect of remote loads upon the maximum shears is relatively less than in the case of moments.

216. Moments in Beams of Two and Three Equal Spans; Uniform Loads.—Inasmuch as the conditions for a theoretical maximum are more likely to occur in beams of two or three spans than where the number of spans is large, an exact analysis will be made of maximum moments at all points for the beam of two spans and for the beam of three spans. The spans will be assumed equal and the beam considered as continuous but freely supported at all points. Assume the dead load to be a uniformly distributed load, $=w$ per lineal foot, and the live load to be also a uniform load, $=p$ per lineal foot, but distributed over such portions of the beam as to cause a maximum moment at the given section. The live-load moments have been calculated for such position of the load as to cause the maximum moment at all points. The results are shown in Figs. 6 and 7. Dead-load moments are given by the dotted lines, live load by full lines. The ordinates as plotted are the coefficients of the quantities wl^2 and pl^2.

The coefficients of wl^2 and pl^2 for the maximum positive and negative moments for the two beams are as follows:

	Maximum near Center of Span (+)	Maximum at Support (−)
Beams of two spans (Fig. 6):		
Dead load	.070	.125
Live load	.095	.125
Beams of three spans (Fig. 7):		
Dead load, 1st span	.080	.100
Dead load, 2d span	.025	
Live load, 1st span	.100	.117
Live load, 2d span	.075	

If the dead and live loads are combined into a single unit for the purposes of calculation, the proper coefficient for $(w+p)$

will depend on the relation of dead and live load. If, for example, the dead load is one-third the live load, then there results,

Beam of two spans:

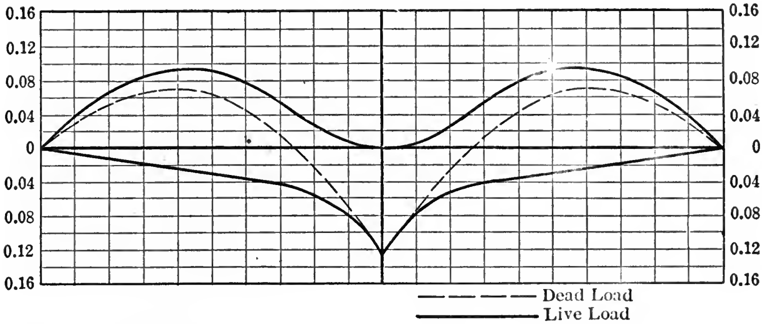
Maximum positive moment $= .089(w+p)l^2$,
Maximum negative moment $= .125(w+p)l^2$.

Beam of three spans:

$$\text{Maximum positive moment}\begin{cases}\text{end span} = .095(w+p)l^2.\\ \text{center span} = .062(w+p)l^2.\end{cases}$$

Maximum negative moment $= .113(w+p)l^2$.

FIG. 6.



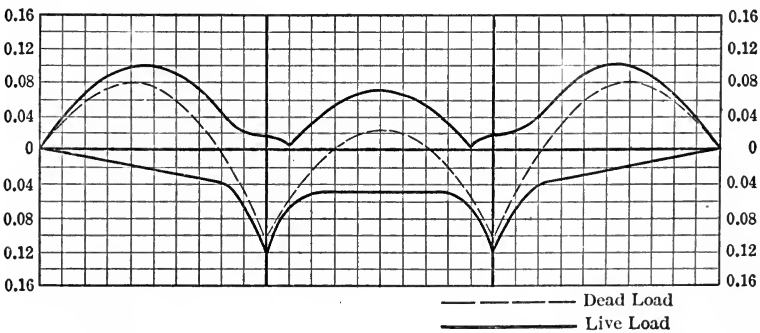
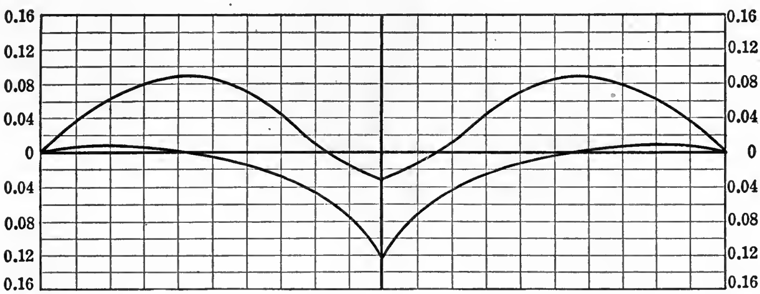


FIG. 7.

In Figs. 8 and 9 the curves show the maximum and minimum moments throughout the beam for the case where $p=3w$, expressed as coefficient of the sum of dead and live load $(w+p)$. These curves are particularly useful in showing the relative

distances from the supports over which positive and negative moments may occur. (See Art. 222.)

FIG. 8.



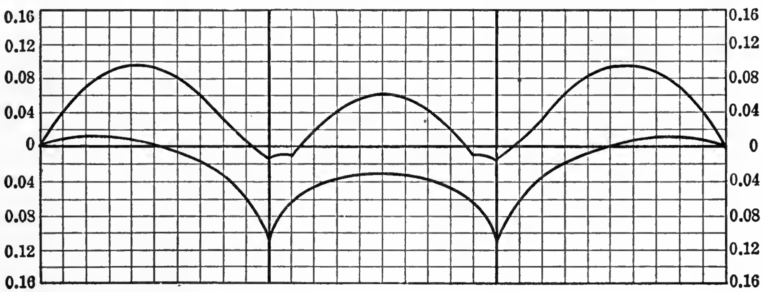


FIG. 9.

217. Moments in Beams of Several Equal Spans; Uniform Loads.—In the case of several spans it will be practically correct

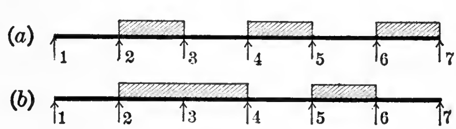


FIG. 10.

in calculating maximum positive moments to consider that the maximum moment at the center of the span is the maximum desired. (Strictly the maximum is generally not quite at the center.) The loading required for maximum live-load moments is illustrated in Fig. 10, which shows in (*a*) the loading for

maximum positive moment in spans 2–3, 4–5, and 6–7, and in (*b*) the loading for maximum negative moment at support No. 3. For the former case each alternate span is loaded and, for the latter, the two adjoining spans are loaded and then each alternate span. Calculations have been made for each span and each support for 4, 5, 6, and 7 spans, and given in Table No. 30. It is found in general that for all spans and supports, except the end span and support adjacent thereto, the maximum positive and negative moments do not vary greatly for the different spans, but for these end spans and supports they are considerably larger than for intermediate spans. The results are, accordingly, arranged in two groups in the table. For the intermediate spans the greatest value for the several spans is given. The table also includes the previous results for two and three spans.

Table No. 30.

COEFFICIENTS FOR MAXIMUM MOMENTS IN CONTINUOUS BEAMS.

No. of Spans.	Intermediate Spans and Supports.				End Span and 2d Support.			
	At Center (+)		At Support (−)		At Center (+)		At Support (−)	
	Dead.	Live.	Dead.	Live.	Dead.	Live.	Dead.	Live.
Two	......	......	......	....	.070	.095	.125	.125
Three	.025	.075	......	....	.080	.100	.100	.117
Four	.036	.081	.071	.107	.071	.098	.107	.120 (.115)
Five	.046	.086	.079	.111 (.106)	.072	.099	.105	.120 (.116)
Six	.043	.084	.086	.116 (.106)	.072	.099	.106	.120 (.116)
Seven	.044	.084	.085	.114 (.106)	.072	.099	.106	.120 (.116)

The quantities in parentheses are the coefficients for live-load moments over supports where the two adjoining spans only are loaded. The effect of loading each alternate span in

addition to these two spans is seen to be small; and considering that such a loading would be extremely improbable, and also the fact that a comparatively small amount of load on the other spans would neutralize this effect, it is apparent that the quantities in parentheses may be taken as reasonable maximum values. The two-span beam should preferably be treated as a special case.

Finally, leaving out of account the two-span beam, the following values may be taken as reasonable maximum values for beams of any number of spans.

	Intermediate Spans.		End Spans.	
	At Center.	At Support.	At Center.	At Support.
Dead-load moments........	.045	.085	.075	.105
Live load moments.........	.085	.015	.100	.115

218. Working Coefficients for Moments for Equal Spans.—It is generally convenient to adopt some simple fraction, such as $\frac{1}{8}$, $\frac{1}{10}$, or $\frac{1}{12}$ for the coefficient for both dead and live loads for ordinary calculations, and if the same coefficient can be used for both dead and live load it is desirable to do so. The live load will generally range from two to five times the dead load. The average coefficients for the combined loads, for various ratios of live to dead load, using the separate values above given, are as follows:

Ratio of Live : Load.	Intermediate Spans.		End Spans.	
	At Center.	At Support.	At Center.	At Support.
2 : 1	.072	.098	.092	.112
3 : 1	.075	.100	.094	.112
4 : 1	.077	.101	.095	.113
5 : 1	.078	.102	.096	.113

It will be seen from this table that for ordinary proportions a single coefficient may well be used for both dead and live loads.

Before adopting final values consideration should be given to certain modifying influences. The beams and slabs are not freely supported as assumed, but are, to a considerable extent, fixed at the supports. This tends to reduce the maximum moments. The supports, also, are of considerable width, so that if the span lengths be taken center to center, the negative moment at the edge of the support is considerably less than the calculated maximum. Thus if the width of support is $\frac{1}{12}$ the span length the negative moment at edge of support is about 25% less than at center of support. The slab also is greatly strengthened by the adjoining floor structure, as explained in Art. 225, and is also much simpler in design than the beam, and hence need not be so liberally proportioned. Furthermore, it is generally convenient to use the same amount of steel over the support as at the center, so that the moment at support will govern the design. Considering all these elements the following coefficients are proposed for both dead and live loads, and for both positive and negative moments:

For slabs of medium or short span:
  Intermediate and end spans . . . . $\frac{1}{12}$

For beams and for slabs of long spans:
  Intermediate spans. . . . . $\frac{1}{12}$
  End spans . . . . $\frac{1}{10}$

The Joint Committee recommendations correspond to the above values with the additional recommendation of $\frac{1}{10}$ for beams of two spans and $\frac{1}{16}$ to $\frac{1}{12}$ for the end moments of beams built into columns.

219. Moments in Beams of Unequal Spans; Uniform Loads.—Where the spans are quite unequal the use of standard coefficients is unsatisfactory, especially for shorter spans. It will often happen that negative moments must be provided for

throughout the entire length of such spans, as in the case of the center span of a three-span girder, Fig. 9, and some estimate of such moments must be made. A sufficiently exact solution of this problem can be made by considering only three or four of the spans of the girder as a continuous structure, thus neglecting the effect of the more remote spans upon the moment in question. The small relative effect of loads on remote spans is well shown in Fig. 5. Thus, for example, to determine the moment at the third support, Fig. 10, consider the four spans 1–5 only, assuming the girder simply supported at 1 and 5. For the maximum positive moment in span No. 3, consider only the three-span girder 2–5, etc. By including one span each side of the one in question for center moments and two supports each side for moments at supports, the results will be sufficiently accurate

This method is very easy of application by the use of general formulas for moments in two- and three-span girders which are readily derived.

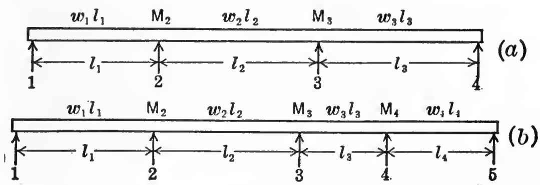


FIG. 11

Girder of Three Spans (Fig. 11*a*).—In the theorem of three moments for uniform loads, Art. 215, let $a = -\frac{1}{4}wl^3$ in general, using a_1 for span 1, a_2 for span 2, etc. Then applying the theorem to spans 1 and 2, and then to spans 2 and 3, and solving for M_2 and M_3, we derive the general formulas:

$$M_2 = \frac{2(a_1+a_2)(l_2+l_3)-(a_2+a_3)l_2}{4(l_1+l_2)(l_2+l_3)-l_2^2}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$M_3 = \frac{2(a_2+a_3)(l_1+l_2)-(a_1+a_2)l_2}{4(l_1+l_2)(l_2+l_3)-l_2^2}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Girder of Four Spans (Fig. 11*b*).—By applying the theorem successively to spans 1 and 2, 2 and 3, and 3 and 4, we derive the general formulas:

$$M_2=\frac{2(a_1+a_2)(l_2+l_3)(l_3+l_4)-(a_1+a_2)\frac{l_3^2}{2}-(a_2+a_3)l_2(l_3+l_4)+(a_3+a_4)\frac{l_2l_3}{2}}{4(l_1+l_2)(l_2+l_3)(l_3+l_4)-l_2^2(l_3+l_4)-l_3^2(l_1+l_2)} \quad (8)$$

$$M_3=\frac{2(a_2+a_3)(l_1+l_2)(l_3+l_4)-(a_1+a_2)l_2(l_3+l_4)-(a_3+a_4)l_3(l_1+l_2)}{4(l_1+l_2)(l_2+l_3)(l_3+l_4)-l_2^2(l_3+l_4)-l_3^2(l_1+l_2)}. \quad . \quad . \quad . \quad (9)$$

The value of M_4, if desired, can be obtained by changing subscripts in eq. (8).

For live-load calculations the proper spans should be loaded and the corresponding values of a used in the formula, the other values being omitted.

220. Example.—Required the dead-load and maximum live-load moments in the continuous girder of Fig. 12 for the following data·

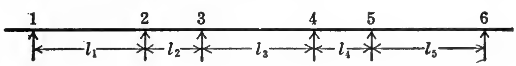


FIG. 12.

$l_1=l_3=l_5=20$ ft.;
$l_2=l_4=10$ ft.;
Dead load $=400$ lbs. per ft.;
Live load $=1000$ lbs. per ft.

The moments at supports 2, 3 and 4 and at centers of spans 1, 2 and 3 will be determined.

M_2. Consider the girder 1–4 as a three-span girder and use eq. (6). For convenience in calculation take the unit of length as 10 feet, thus making the span lengths 2 and 1 respectively. The resulting moments will then be multiplied by $10^2=100$ to get values in foot-pounds. Then for the short spans $a_2=a_4=-\frac{1}{4}wl^3=-\frac{1}{4}w\times100$, and for the long spans $a_1=a_3=-\frac{1}{4}wl^3=-\frac{1}{4}w\times8\times100$. Then from eq. (6)

$$M_2=-\tfrac{1}{4}w\frac{2(8+1)(1+2)-(1+8)}{4(2+1)(1+2)-1}\times100=-\frac{9}{28}w\times100=-12,880 \text{ ft.-lbs.}$$

For live-load moment, use eq. (6) and omit a_3 (third span not loaded).

$$M_2=-\tfrac{1}{4}p\frac{2(8+1)(1+2)-1\times1}{35}\times100=\frac{53}{140}p\times100=-37,900 \text{ ft.-lbs.}$$

M_3. Consider the four-span girder 1–5, and for live-load moment load spans 2 and 3. Use eq. (9). There results the values:

$$\begin{aligned}\text{Dead-load moment} &= -\ 8{,}720 \text{ ft.-lbs.}\\ \text{Live-load moment} &= -31{,}400 \text{ ft.-lbs.}\end{aligned}$$

M_4. By reason of symmetry $M_4 = M_3$.

Center Moment in Span 1.—Consider the three-span girder 1–4 and load spans 1 and 3. The values of M_2 are for dead load, $-12{,}880$ ft.-lbs., and for live load $-\frac{2}{7}p = -28{,}600$ ft.-lbs. For the center point the positive moment is found from the general formula, eq. (5) $M_c = M' - \frac{M_1 + M_2}{2}$. In this case $M_1 = 0$, and $M' = \frac{1}{8}wl^2$ or $\frac{1}{8}pl^2$. Then for dead load, $M_c = \frac{1}{8} \times 400 \times 20^2 - \frac{12880}{2} = +13{,}560$ ft.-lbs. and for live load $M_c = \frac{1}{8} \times 1000 \times 20^2 - \frac{28600}{2} = +35{,}700$ ft.-lbs.

Center Moment in Span 2.—Consider the girder 1–4 and load 2 only. The values of M_2 and M_3 are

$$M_2 = M_3 = \begin{cases}\text{dead oad} = -12{,}880 \text{ ft.-lbs.}\\ \text{live load} = -3570 \text{ ft.-lbs.}\end{cases}$$

The positive moment at the center will then be

$$\begin{aligned}M_c, \text{ dead load} &= \tfrac{1}{8} \times 400 \times 10^2 - 12880 = -7880 \text{ ft.-lbs.,}\\ M_c, \text{ live load} &= \tfrac{1}{8} \times 1000 \times 10^2 - 3570 = +8900 \text{ ft.-lbs.}\end{aligned}$$

The negative moment will be found by loading spans 1 and 3. The resulting live-load moment $M_2 = M_3 = -28{,}500$ ft.-lbs. There being no load on the span the live-load moment at the center is also $-28{,}500$ ft.-lbs. The center moments are then

$$\begin{aligned}M_c, \text{ dead load} &= -7880 \text{ ft.-lbs.,}\\ M_c, \text{ live load} &= -28{,}500 \text{ ft.-lbs.}\end{aligned}$$

Center Moment in Span 3.—Consider the girder 2–5 and load span 3. Use eq. (6) for M_3. $M_4 = M_3$. The results are

$$\begin{aligned}M_3, \text{ dead load} &= -11{,}250 \text{ ft.-lbs.,}\\ M_3, \text{ live load} &= -25{,}000 \text{ ft.-lbs.}\end{aligned}$$

Then the center moments are:

$$\begin{aligned}M_c, \text{ dead load} &= \tfrac{1}{8} \times 400 \times 20^2 - 11{,}250 = +8750 \text{ ft.-lbs.,}\\ M_c, \text{ live load} &= \tfrac{1}{8} \times 1000 \times 20^2 - 25{,}000 = +25{,}000 \text{ ft.-lbs.}\end{aligned}$$

This gives a total positive moment of 33,750 ft.-lbs., compared to 49,260 ft.-lbs. for span 1.

The maximum negative live-load moment in span 3, loads on spans 2 and 4, is found to be −3100 ft.-lbs., which is less than the positive dead-load moment.

The foregoing results are collected in the following table:

	Dead.	Live.	Total.
M_2	−12,880	−37,900	−50,780
M_3	− 8,720	−31,400	−40,120
M_c, span 1	+13,560	+35,700	+49,260
M_c, span 2	− 7,880	{ + 8,900 −28,500	{ + 1,020 −36,380
M_c, span 3	+ 8,750	+25,000	+33,750

If it is attempted to estimate moments by means of coefficients it would probably be reasonable to apply a coefficient of 1/10 for M_2 and 1/12 for M_3, using the full 20-foot span. This gives a value for total moment of $M_2=1/10\times1400\times20^2=56{,}000$ ft.-lbs. and $M_3=1/12\times1400\times20^2=46{,}700$ ft.-lbs., both values being considerably larger than the results here calculated. Still larger differences are noted for the center moments of spans 1 and 3, using the same coefficients, 1/10 and 1/12.

221. Moments in Beams, Concentrated Loads.—In most cases of continuous beams the loading may be assumed as uniformly distributed over a span, but in some cases the effect of heavy concentrated loads must be considered for which the uniform load method or the use of coefficients may not lead to satisfactory results. To obtain more exact results requires, in general, the application of the theorem of three moments (Art. 215) for concentrated loads, using influence lines if necessary for live-load effects.

A common case of concentrated loads is that where girders receive their loads from cross-beams arranged with two, three, or four to a span (Fig. 13). In this case the beams are generally spaced equally and arranged symmetrically in the span, and for any one span the applied loads are equal. For a group of symmetrically spaced loads the quantities $\Sigma(k-k^3)$ and $\Sigma(2k-3k^2+k^3)$ of eq. (1*b*) are equal, so that for equal loads

in the span the second member of this equation may be written $-P_1l_1^2\Sigma(k_1-k_1^3)-P_2l_2^2\Sigma(k_2-k_2^3)$, in which P_1 is the load applied at each panel point in span 1 and P_2 the load in span 2. These quantities may be represented by a_1 and a_2, as in the case of uniform loads, and the theorem applied exactly as in that analysis. The formulas of Art. 219 may therefore be applied directly to this special case of concentrated loads, substituting for the several values of a the corresponding quantities $-Pl^2\Sigma(k-k^3)$ from the spans in question.

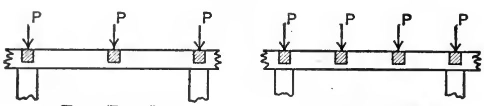


FIG. 13.

The problem may be still further simplified by calculating the quantity $\Sigma(k-k^3)$ for several common cases, say for 2, 3, 4 and 5 panels per span. In each case the load is P and a beam or load comes directly over each support. For two panels $k=\frac{1}{2}$ and $\Sigma(k-k^3)=\frac{3}{8}$. For three panels $k=\frac{1}{3}$ and $\frac{2}{3}$, and $\Sigma(k-k^3)=\frac{2}{3}$. For four panels $k=\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$, and $\Sigma(k-k^3)=\frac{15}{16}$, and for five panels $\Sigma(k-k^3)=\frac{6}{5}$. Let wl = total load per span in all cases (w = average load per unit length); then for two panels $P=wl/2$, for three panels $P=wl/3$, for four panels $P=wl/4$, and for five panels $P=wl/5$. Substituting these values for $\Sigma(k-k^3)$ and for P, we have the following values of the term $a=-Pl^2\Sigma(k-k^3)$ for the different cases:

$$
\begin{aligned}
&2 \text{ panels, } a=-\tfrac{3}{16}wl^3;\\
&3 \text{ panels, } a=-\tfrac{2}{9}wl^3;\\
&4 \text{ panels, } a=-\tfrac{15}{64}wl^3;\\
&5 \text{ panels, } a=-\tfrac{6}{25}wl^3.
\end{aligned}
$$

These values of a, expressed in terms of w, or average load per unit length, are closely equal to the value of $-\frac{1}{4}wl^3$ for a

uniformly distributed load of w per unit length. They are, however, all somewhat smaller, the differences being as follows:

Amount.	Per Cent of $\frac{1}{4}wl^3$.
2 panels, $\frac{1}{16}wl^3$	25
3 panels, $\frac{1}{36}wl^3$	11
4 panels, $\frac{1}{64}wl^3$	6
5 panels, $\frac{1}{100}wl^3$	4

It follows, therefore, that for equal spans the values of all moments *at supports* where the loads are applied at panel points are smaller than the moments for the same load *uniformly distributed* as heretofore calculated by the percentages given in the foregoing table.

The positive moments at centers of spans are in some cases increased and in other cases reduced slightly by the panel arrangement, but the percentage change cannot be given in general terms. The positive moment at the center of the span is by eq. (5) $M_c = M_c' - \dfrac{M_1 + M_2}{2}$. The values of M_1 and M_2 are reduced by the amounts heretofore given, but M_c', the moment in a simply supported beam, is reduced only where an odd number of panels is used. For an even number of panels it has the same value as for a uniformly distributed load. For three panels $M_c' = Pl/3 = \frac{1}{9}wl^2$, and for 5 panels $M_c' = \frac{3}{5}Pl = \frac{3}{25}wl^2$. For two and four panels $M_c' = \frac{1}{8}wl^2$.

Table No. 31, similar to Table No. 30, gives coefficients of wl^2 for 2, 3, and 5-span girders for 2, 3, and 4 panels per span, w being the average load per foot of span. Comparing these with the values given in Table No. 30 it is seen that excepting for the two-span girder, the differences are not great, but the moments at center and at supports are more nearly equal than in the other case. For a three-panel arrangement the moment calculated by the use of the general coefficients of $\frac{1}{10}$ and $\frac{1}{12}$ may be well reduced by 10%. For a larger number of panels or for an even panel arrangement no material reduction can be made.

Table No. 31.

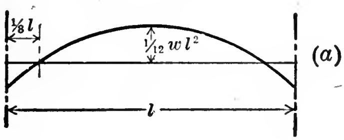
COEFFICIENTS OF wl^2 FOR MAXIMUM MOMENTS IN CONTINUOUS BEAMS, LOADS CONCENTRATED AT PANEL POINTS.

No. of Spans.	Intermediate Spans and Supports.				End Span and 2d Supports.			
	At Center (+)		At Support (—)		At Center (+)		At Support (—)	
	Dead.	Live.	Dead.	Live.	Dead.	Live.	Dead.	Live.
Two panels:								
Two......	......	......	......	....	.078	.102	.093	.093
Three.....	.050	.087	......	....	.097	.106	.075	.087
Five......	.065	.095	.060	(.078)	.085	.105	.079	(.087)
Three panels:								
Two......	......	......	......	....	.074	.093	.111	.111
Three.....	.022	.067	......	....	.081	.096	.089	.104
Five......	.041	.076	.070	(.092)	.080	.095	.094	(.103)
Four panels:								
Two......	......	......	......	....	.067	.096	.116	.116
Three.....	.032	.078	......	....	.078	.101	.093	.109
Five......	.051	.088	.074	(.097)	.076	.100	.098	(.109)

222. Length of Rods for Positive and Negative Reinforcement.—In determining the length of rods required for reinforcement, it is necessary to ascertain approximately the variation of bending moment along the beam under the assumed load conditions. For positive moment the span in question is fully loaded and the adjacent spans not loaded. For the dead load the point of zero moment is about $.2l$ from the support, and for live load about $.1l$ therefrom. For the total moment it may be assumed in the case of interior spans that the point of zero moment is $\frac{1}{8}l$ from the support and that the moment curve is a parabola with middle ordinate equal to the assumed maximum moment. The moment curve will then be as shown in Fig. 14*a*. For negative moment the maximum will occur when both adjacent spans are loaded, and the live-load moment diagram for these spans

will be about as shown in Fig. 14*b*. The moment at the middle support B may be taken at $\frac{1}{10}wl^2$ and at supports A and C at about $.03wl^2$. With these values the point of zero moment will be $.22l$ from B. For dead load the point of zero moment is $.2l$ from B. It may then be assumed that for equal panels the point of zero moment is about $\frac{1}{4}l$ from the support, and that the value of the moment varies as a straight line from this point to the support.

The foregoing analysis applies to the case of maximum negative moment at supports and the corresponding loading. It has been seen, however, from Fig. 9 and the problem of Art. 220 that it is possible to have negative live-load moment through-



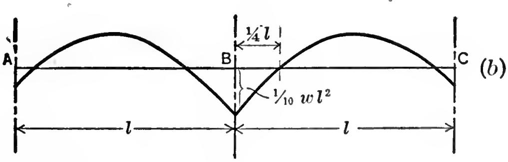


FIG. 14.

out any given span by loading the spans on each side of the one in question, which will be greater than the positive dead-load moment. In the case of short spans flanked by long ones this negative moment may be large and should be provided for by suitable reinforcement throughout the beam. For spans approximately equal in length, negative reinforcement carried to the quarter points, or, in case of heavy loading, to the third points, will be sufficient. A very considerable negative moment can be carried by the concrete in tension, especially in the case of T-beams, and any deficiency in strength along the central part of the span would result only in a slightly greater positive moment being thrown upon the loaded spans.

223. Effect of Variable Moment of Inertia on Bending Moments.—It has been assumed in the determination of bending moments in the foregoing articles that the moment of inertia of the beam is uniform throughout. As there shown, the resulting maximum moments at center and support are not greatly different and for all practical purposes may be taken as equal, so that if fully reinforced the amount of steel and the moments of inertia will accord with the assumption. It is the practice of some designers, however, to consider the beam primarily as a simple beam and design it to carry all, or nearly all, of the load as such. A relatively small amount of steel is then placed in the top of the beam over the support, mainly to prevent objectionable cracks, but which is also in some cases counted upon to carry a portion of the moment. It is therefore of importance to determine the actual moments and stresses which occur at the center and support for various proportions of reinforcements.

This problem has been concisely analyzed by Mr. P. E. Stevens * for a beam fixed at the ends, and the following results are from his paper. He assumed a uniform moment of inertia, $=I_0$, for that portion over the support in which negative moments exist, and another moment of inertia, $=I_1$, for that portion of the central part of the span in which positive moments exist (see Fig. 15). Let M_0= bending moment at support and M_1=

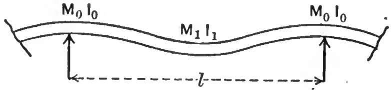


FIG. 15.

bending moment at center. Then for a uniformly distributed load the ratios of these bending moments for various ratios of $I_0 : I_1$ are as given in the table below.

* Trans. Am. Soc. C.E., Vol. LX, 1908, p. 496.

MOMENTS AND STRESSES IN BEAMS WITH VARIABLE MOMENT OF INERTIA.

Ratio. $I_0 : I_1$	Ratio. $M_0 : M_1$	$\frac{M_1}{\frac{1}{8}wl^2}$	$\frac{M_0}{\frac{1}{8}wl^2}$	Ratio of Stresses, $f_0 : f_1$
$\frac{1}{5}$	0 8	.55	.45	4 1
$\frac{1}{4}$	0.9	.52	.48	3.7
$\frac{1}{3}$	1.1	.48	.52	3.2
$\frac{1}{2}$	1 4	.42	.58	2.8
1	2 0	$.33\frac{1}{3}$	$.66\frac{2}{3}$	2.0
$1\frac{1}{2}$	2 6	.28	.72	1.7
2	3 0	.25	.75	1.5
6	5 8	.15	.85	1.0

The table also gives in the third and fourth columns the values of these moments expressed as percentages of the sum of the center and end moments, $\frac{1}{8}wl^2$. In the last column are given the ratios of the corresponding unit stresses in the steel, f_0 and f_1, assuming the moments of inertia to be proportional to the steel areas. These results show approximately the actual distribution of bending moments for various proportions of steel. If, for example, the moment of inertia is the same throughout ($I_0/I_1 = 1$) the center moment is $\frac{1}{3}(\frac{1}{8}wl^2)$ and the end moment $\frac{2}{3}(\frac{1}{8}wl^2)$, as is well known. For these conditions the ratio of steel stresses will be the same as the moments $= 2 : 1$. Again, suppose the amount of steel at the center be based on a moment $\frac{1}{10}wl^2$, and that one-half as much steel be placed at the end. Then $I_0/I_1 = \frac{1}{2}$ and $M_0/M_1 = 1.4$, and the ratio of unit stresses $f_0/f_1 = 2.8$. The actual moment at the center will then be 42% of $\frac{1}{8}wl^2 = .052wl^2$ instead of $.10wl^2$, as assumed, and the actual unit stresses will be 52% of the assumed. The moment at the end will be 58% of $\frac{1}{8}wl^2$, and the unit stress 2.8 times that at the center, or 1.45 times the assumed working value. The steel at the center will therefore be under-stressed and that at the end over-stressed. If a small amount of steel be placed at the end, such that $I_0/I_1 = \frac{1}{5}$, and the full bending moment provided for at the center, the stress at the center will be 55% of the work-

ing stress and at the end will be $4.1 \times .55 = 2.25$ times the working stress.

From this analysis it is seen that if a small amount of steel is placed over the supports this steel will be over-stressed and that at the center under-stressed. Applying these results to a continuous girder supporting a moving load, it will be found that the positive and negative moments will be more nearly equal than indicated in the above table, but the general effect of a variation in moment of inertia and amount of steel will be similar to that above given.

224. Shears in Continuous Beams.—The maximum shears near supports are not greatly affected by moving loads. For intermediate spans the maximum end shear may be taken at one-half of the span load; for end spans the shear near the second support will be approximately six-tenths of a span load.

Where the load is applied at panel points the shears will be reduced by approximately that proportion of the load applied directly over the support—one-half for two panels, one-third for three panels, one-fourth for four panels, etc.

225. Effect of Rigid Supports on the Resisting Moment.—If a flat slab is held between unyielding supports, such as fixed I-beams, a strength, or resisting moment, will be developed in the slab even though there be no steel reinforcement. Failure cannot take place without the crushing of the concrete either at the center or at the support. For short spans the resisting moment (the so-called "arch action") is about as great as will exist in the slab if reinforced and simply supported at the ends. In the case of a flat reinforced slab such rigid supports likewise add considerably to the strength of the slab, giving the effect of partial continuity.

In practice, the supports of slabs of short span length, whether consisting of I-beams or of concrete beams of which the slab is a part, are rendered very rigid by reason of the action of the adjoining floor-panels. Even where the slabs are simply supported on the tops of steel beams the adjoining

slabs prevent to some extent lateral motion, rendering all such spans partially continuous. The strengthening effect of rigid supports is, therefore, especially great in the case of narrow floor-spans and where there is a large number of consecutive unbroken panels. Under such conditions reinforcement against negative moment is hardly necessary. For long spans and for spans on the outside of a system the effect is small.

226. Table of Safe Loads for Floor Slabs.—Table No. 34, Chapter XIV, gives the span lengths for floor slabs for various live loads per square foot, and for various values of working stresses f_s and f_c. These tables have been calculated for bending moments of $\frac{1}{8}wl^2$ and $\frac{1}{12}wl^2$ and for $n=15$. For $M=\frac{1}{10}wl^2$ the safe span may be found by direct interpolation.*

DETAILS OF CONSTRUCTION.

227. Design of Floor Slabs.—*Floor slabs Supported on Steel Beams.*—Many "systems" have been developed of this type of construction, differing from each other in form of steel used, position of the concrete relative to the beam, use of curved or flat slabs, use of various kinds of hollow tile in connection with the concrete, etc. In building construction where the framework is steel, the concrete is generally arranged to act as fireproofing material as well as floor support.

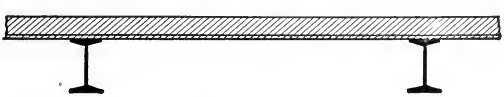


Fig. 16.

Fig. 16 shows the floor placed directly on the tops of the beams. The reinforcement may be small rods or a meshwork of expanded metal or woven fabric. If reinforced as shown, the slab must be calculated as a simple beam, there

* The exact value of the safe span length for $M=\frac{1}{10}wl^2$ is 11.8% greater than for $M=\frac{1}{8}wl^2$, whereas by interpolation we get a span length 11.3% greater.

being no reinforcement against negative moment over the support. For spans of considerable length some reinforcement for negative moments should be used to secure economy and to prevent cracks in the upper surface, although the lateral rigidity due to adjoining panels is of much assistance, as explained in Art. 225. Fig. 17 shows a more common design of non-continuous slab, the concrete being supported on the lower flange and the entire beam fire-proofed.

FIG. 17.

Fig. 18 shows a standard form of construction in which the slab is practically continuous. The reinforcing material may be rods or a metal fabric continuous over several spans.

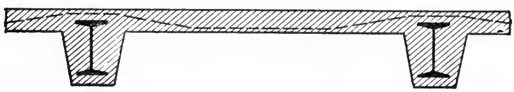


FIG. 18.

228. *Floor-slabs in All-concrete Construction.*—Wherever concrete beams are used the slab and beam are usually built simultaneously, giving a monolithic structure. The slab thus constitutes part of the beam, but to be effective these two parts must be well tied together. Where cross-beams are used the span of the slab will commonly range from 4 to 7 feet in length. For very short spans a reinforcement of rods or metal meshwork near the bottom only will be effective as explained in Art. 225. This reinforcement, if of rods, should be laid with lapped and broken joints to give continuity and to prevent the localization of contraction cracks in undersirable places (Fig. 19). The beam, if well bonded to the slab, will make a very rigid support comparable to the I-beam.

In the case of spans longer than 4 or 5 feet it becomes desirable to reinforce against negative moment. This may readily

be done by bending up a part or all of the rods and extending the bent ends beyond the beam (Fig. 20). In (*a*) each rod fur-

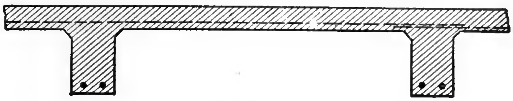


Fig. 19.

nishes one bottom and one top reinforcement; in (*b*) the rods are longer, each one furnishing two bottom and two top reinforcements.

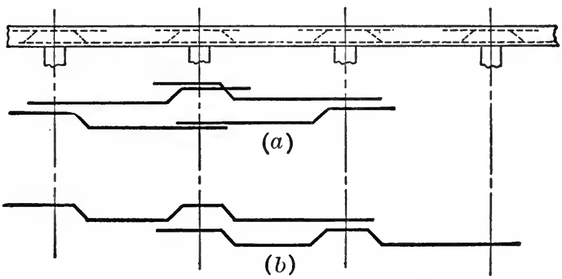


Fig. 20.

The result may also be arrived at by using separate straight rods, as shown in Fig. 21. The plan of bent rods has a slight advantage as it reinforces somewhat against shearing failures, but this is not usually important in slabs. For very heavy loads, however, it becomes of importance, and the same care should be used as in the design of large beams.

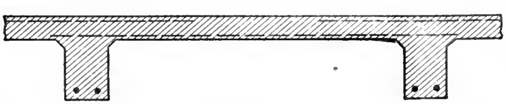


Fig. 21.

229. Reinforcement to Prevent Cracks.—While longitudinal reinforcement of oblong slabs is of little value in carrying loads, a small amount is nevertheless often desirable in preventing cracks and in binding the entire structure together. For a

close beam spacing such reinforcement is hardly necessary, as the beam reinforcement itself thoroughly ties the structure longitudinally along the beam lines. For wide beam spacing it is more important. Just what amount of steel is needed is a matter of experience. The use of $\frac{1}{4}$-inch or $\frac{3}{8}$-inch rods spaced about 2 feet apart is common practice. If a metal fabric is used for oblong panels, the longitudinal metal should be proportioned in accordance with the principles discussed in this and the preceding articles.

230. Slabs Made of Concrete and Hollow Tile.—For comparatively long spans and light loads a form of construction is often employed in which hollow tile are embedded in the concrete slab so as to reduce the volume of concrete without greatly re-

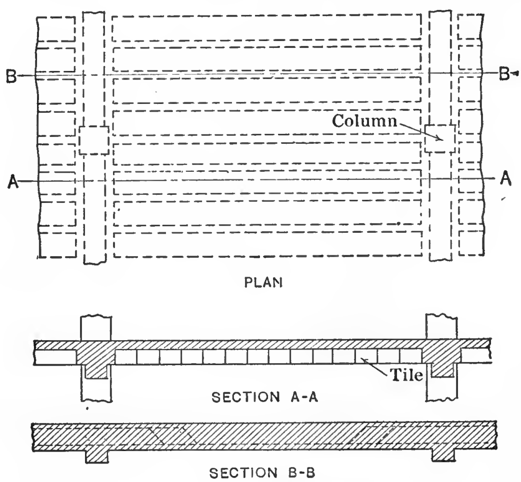


Fig. 22.

ducing its resisting moment. Fig. 22 illustrates this type of construction. The concrete forms virtually a series of small T-beams the stems being separated by the tile. Each T-beam is reinforced by one or two small rods. The tile thus serve as forms and give a uniform thickness to the slab with a good plastering surface below. No allowance is made for the tile in calculating strength.

231. Beams and Girders.—*Economical Arrangement.*—The arrangement of columns, girders, and beams is determined according to the same principles as in steel construction. The spacing of columns and girders will be determined largely by architectural considerations. The best spacing of cross-beams will differ in different cases. Where the spacing of girders is not too great (12 to 15 feet) and where cross-beams are not needed to secure lateral stiffness, it will be a question of omitting all cross-beams, of inserting them only at columns so as to form a square or nearly square panel, or of spacing them at closer intervals of 4 to 8 feet, using two or more to a girder panel. The analysis of Chapter VIII shows that double reinforcement will not be economical for oblong panels. Cross-beams, if used, should therefore be arranged to give very nearly square panels or else be spaced much more closely, designing the reinforcement so as to carry the entire load to the beams and thence to the girders.

If not otherwise needed, the use of cross-beams to secure square panels effects little if any saving. The amount of concrete will be less, but the amount of steel required will be more, and the extra beam will be more costly per unit volume than the slab. However, for the sake of lateral stiffness it will usually be desirable to place cross-beams at columns.

Where close spacing of beams is adopted the best arrangement depends upon the loading and the working stresses, as well as upon the cost of the material and forms. Heavy loads and low stresses call for large weights of concrete and tend to require the use of the material more in the form of deep ribs or beams, as the deeper the beam the greater its moment of resistance for a given volume. If cross-beams are used, a spacing greater than 10 or 12 feet or less than 4 or 5 feet will seldom be economical. Architectural considerations will often govern, and frequently building regulations relative to ratio of span to depth will control.

232. *Design of Beams.*—In the design of beams the chief features are the determination of the cross-section, the amount

of steel and its make-up, provision for shearing stress, provision for negative bending moment and connections with slabs, other beams, and columns. The proportions of the beam, whether considered as a rectangular beam or as a T-beam, will be determined by considerations discussed in Chapter V. Ratios of depth to width greater than 2 or $2\frac{1}{2}$ are seldom used. Requirements of head-room, space for rods, and shearing strength will limit the possible variations in proportions to a comparatively narrow range. Deep beams are economical of concrete but cost more for forms than do shallow beams.

If the beam may be calculated as a T-beam, the width of slab which may be counted on as a part of the beam is an important question. Specifications usually allow a width of six to ten times the thickness of the slab, but not to exceed the width between beams. As regards *strength* it would be very difficult to secure so thorough a reinforcement of web as to make it possible to crush a flange as much as four times the width of the web; the excessive shearing stresses in the web would cause failure. As regards *stiffness,* which controls the position of the neutral axis, the width of the slab to be counted as part of the beam may and should be taken relatively great. The width of flange being known, the design of the T-beam consists chiefly in the design of the web and the calculation of the steel cross-section. It will be only in the case of large girders that the compression stress in the concrete will be a determining factor. Usually there is a large excess of material.

If the beam is to be considered as continuous over supports, the moment of resistance at the support must also be investigated. At this point the tension side is uppermost and the effective beam is now a *rectangular* beam. The maximum moment is about the same as at the center, thus requiring about the same amount of steel at the top as is required in the center of the span at the bottom. The maximum compression in the concrete will be greater than in the center and will probably determine the size of beam required unless special provision is made for these stresses. This may be done

by increasing the depth of the beam near the end, as shown in Fig. 23, or by the use of compression reinforcement. Such reinforcement may be provided to a considerable extent by merely continuing the horizontal steel sufficiently to give the necessary bond strength. If the horizontal steel near the end amounts to as much as 1% of the rectangular section, then considering both of two adjoining beams there would be available about 2% of compressive reinforcement. By Plate IX, Chapter XIV, this amount would reduce the compressive concrete stresses about 40%. This would usually be sufficient. Inasmuch as a slight excess of stress at this point does not in any way endanger the structure, merely increasing somewhat the positive moment on the beam, it would seem to be proper to permit the use of a higher working stress than at the center.

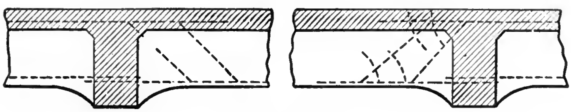


Fig. 23.

An increase of 10–15% would be entirely safe. The necessary top steel at the end may be provided, as in the slab, by bending up a portion of the lower rods, or by using separate short rods, or by both methods combined. To provide thoroughly for negative moment the upper reinforcement should extend to about the third point, and in some cases still farther.

The treatment of girders is the same as described for beams, it being especially important that the reinforcement pass well through the column.

The arrangement of shear or web reinforcement for beams and girders is of great importance, as it is in these forms where the web tensile stresses will be high. At points where the allowable shearing stress in the concrete is exceeded steel must be added in some form to carry a part of the stress. Where bent-up rods are used, as in Fig. 23, these rods aid greatly in carrying shear, and where not spaced too widely may be counted on to add perhaps 50% to the strength of the web. For thorough

web reinforcement the stirrup is usually employed, or some form of bent bar closely spaced. This reinforcement may be calculated as explained in Art. 141, not too much reliance being placed on one or two bent rods. Web reinforcement will usually be needed only for the end quarter or third of the beam. Near the support, where the moment is negative, the tendency is for diagonal cracks to start at the top, while farther along the cracks tend to start at the bottom, as shown in Fig. 23. Stirrups at points of negative moment should loop about the upper bars, and at points of positive moment should loop about the lower bars. A correct understanding of the diagonal stresses in such continuous beams is important.

The beam should be well bonded to the slab, especially near the end where the differential stresses between the two parts are large. This is well accomplished by means of the bent rods brought up as high as possible, and by means of the

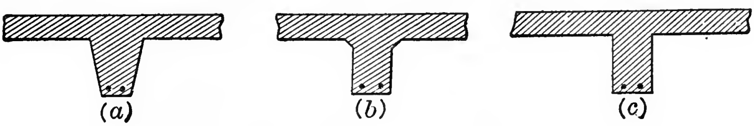


Fig. 24.

slab reinforcement which crosses the beam. Along the center of the beam the matter is not of so great importance, but it is better to provide such bond by some form of vertical reinforcement, such as stirrups, extending up into the slab at occasional intervals. This is of especial importance in the case of girders where the main slab reinforcement runs parallel to the beam. A good bond is also more necessary the thinner the sections. Sections shown in Fig. 24, (*a*) and (*b*), are more favorable than such a section as in Fig. 24 (*c*). Sharp reentrant angles in such a brittle material as concrete are points of weakness, and where they exist a steel bond is desirable.

233. The Unit Frame for Beams.—In executing work a practical difficulty of considerable importance is that of placing and keeping all bars, stirrups, etc., in their proper position until

the concrete is in place. Very considerable labor is required in wiring bars in position, or in providing other means of support, and careful supervision is necessary during construction to see

Fig. 25.

that they remain in place. To avoid these difficulties various arrangements have been devised for fastening together all, or a part, of the rods of a single span into a group which can be handled as a unit. giving rise to the so-called "unit frame." These units

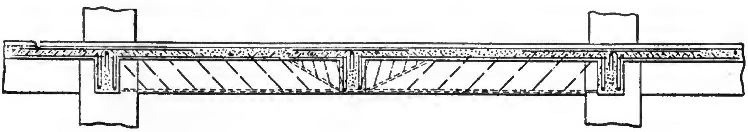


Fig. 26.

are obviously not so adaptable to a great variety of conditions as single independent bars, but their advantages are considerable, and they are being used to quite an extent.

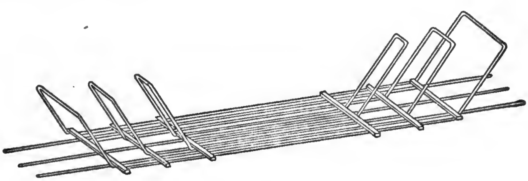


Fig. 27.

The Kahn bar illustrated in Fig. 25 may be considered a form of unit frame, inasmuch as the shear reinforcement is integral with principal tension bar. These shear bars are made by shearing and bending up portions of the flat wings of the original rolled section. Fig. 26 illustrates the use of this bar in beam and girder construction. Fig. 27 illustrates a unit frame

of the Cummings system. The bend rods are formed in loops, thus giving a very satisfactory bond. Figs. 28 and 29 illustrate other forms of unit frames, the former being the type used by the American System of Reinforcing, Chicago, Illinois, and the

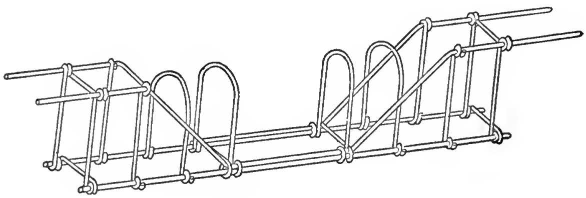


FIG. 28.

latter by the Corrugated Bar Company, Buffalo, New York. All such unit frames are proportioned to suit the particular requirements of the case.

FIG. 29.

234. The Unit System of Construction.—A form of construction which has been developed to a considerable extent makes use of reinforced concrete beams, girders, columns, and other elements of construction which are separately molded at any convenient place and erected into place in much the same manner as steel units. Various details are used to connect the separate units, consisting generally of certain projecting bars fitting into suitable pockets and held in place by pouring in a small amount of concrete or by constructing a certain portion of the structure, such as the slab, after the units are assembled. This type of construction has its greatest advantage where many units of the same dimension may be used, especially if they can be made of moderate size. This construction does not have the strength of monolithic work, and strength against

lateral forces must be provided for principally by means of some form of bracing or by the walls and partitions.

235. Design of a Slab, Beam, and Girder Floor.—To illustrate the principles given in the preceding articles, a complete design will be made of a floor panel of the general dimensions shown in Fig. 30. The floor will be designed for a uniform live load of 200 pounds per square foot. It will be assumed that all members are poured at the same time so that continuous girder conditions will determine the design of slabs, beams, and girders. It will be further assumed that the panel in question is an interior panel, for which, according to Art. 218, the positive moments at the span center and the negative moments at points of support can be taken as $M = 1/12wl^2$.

The concrete will be assumed as of 2000 pounds ultimate strength and working stresses as follows:

$f_s = 16{,}000$ lbs./in.^{2} longitudinal steel, 10,000 lbs./in.^{2} in stirrups;
$f_c = 650$ lbs./in.^{2};
$v = 40$ lbs./in.^{2}, where no web reinforcement is provided, and 100 lbs./in.^{2}, where the web is thoroughly reinforced with bent bars and stirrups not more than $\frac{1}{2}d$ apart;
$u = 80$ lbs./in.^{2} with an allowable increase of 50%, as in Art. 148.

Design of Slab.—From Fig. 30 the span of slab is 6 feet 8 inches or 6.7 feet. Assume a 4-inch slab, which weighs 50 pounds per square foot. Considering a section of slab 1 foot wide, the total load to be carried is $200 + 50 = 250$ pounds per foot of slab, and

$$M = 1/12wl^2 = 1/12 \times 250 \times 6.7^2 \times 12 = 11{,}200 \text{ in.-lbs.}$$

From Plate II with $f_s = 16{,}000$ and $f_c = 650$, we have $R = 108$ and $j = 7/8$. Then $bd^2 = M/R = 11{,}200/108 = 104$. With $b = 12$ inches, $d^2 = 104/12 = 8.65$ and $d = 2.94$ inches. Use $d = 3$ inches, assuming 1 inch of concrete below the steel. The assumed 4-inch slab is therefore satisfactory.

The steel area required at the span center and at points of support $= M/f_sjd = 11{,}200/(16{,}000 \times 7/8 \times 3) = 0.264$ square inch. This area is furnished by $\frac{3}{8}$-inch round rods spaced 5-inch centers.

The general arrangement shown in Fig. 20 (*b*) will be used.

Design of Beams.—Since the beam and slab are cast together, a tee-beam section can be assumed for the portion of the beam under positive moment. At the ends of the beam, where the moment is negative, the tensile stresses are on the top side, and as no tension is assumed as carried

by the concrete, the beam section at supports is rectangular. Both positive and negative moments will be taken at $1/12wl^2$ and their variation assumed in accordance with Art. 222 and as shown in Fig. 31. Thus the section B–F is to be considered as a tee-beam, and section A–C is

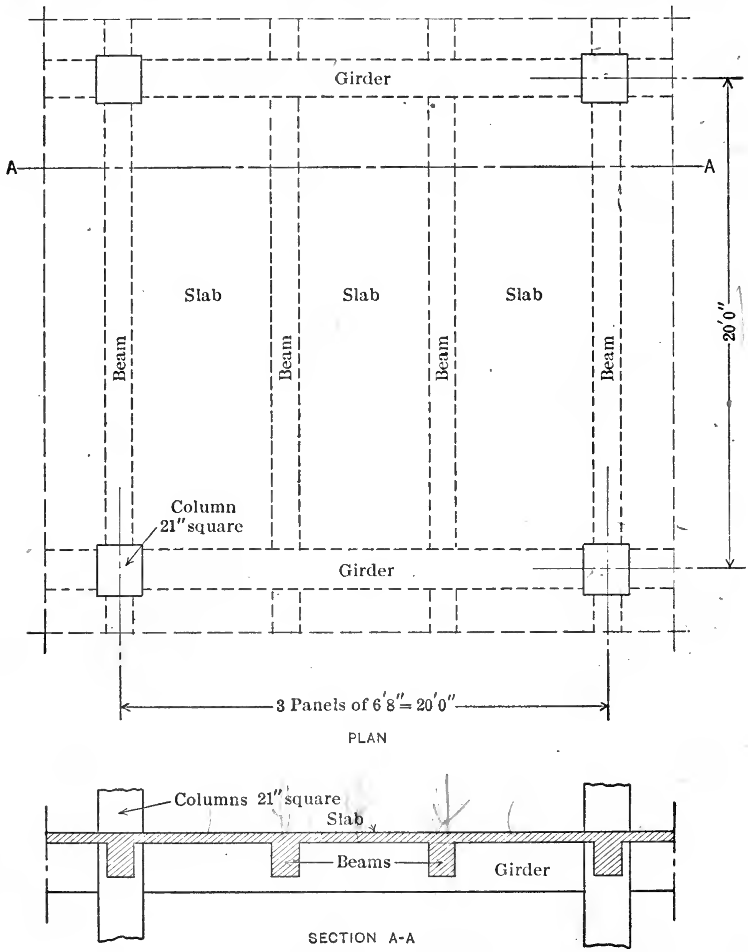


FIG. 30.

to be considered as a rectangular beam. The section B–C must be able to take either positive or negative moment.

As shown in Fig. 30, each beam supports a section of floor which is 6.7 feet wide and 20 feet long. The load carried by each beam is the

live load of 200 pounds per square foot, the slab, which weighs 50 pounds per square foot and the portion of the stem of the beam below the floor slab. We will consider first the tee-beam section from B to F, Fig. 31; the design is similar to that given in Art. 152. The live and slab load as given above is 250 pounds per square foot or $250 \times 6.7 = 1670$ pounds per foot of beam. Assuming a stem 12 inches wide, which will allow four bars to be placed in a single layer, and a total depth of 21 inches, including 2 inches of protective covering below the steel, the weight of stem is $(21-4) \times 12 \times 150/144 = 210$ pounds per foot. The total load to be carried is then $1670 + 210 = 1880$ pounds per foot.

As in Art. 152, the required beam section is determined by the shear at the supports. The value of j to be used in the determination of shear

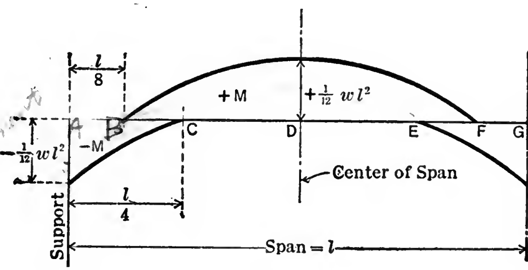


Fig. 31.

area, therefore, depends upon the properties of the double reinforced rectangular section at the end of the beam. As this is not known, a preliminary estimate can be made with $j = 7/8$. This can be revised later if necessary.

For a load of 1880 pounds per foot, the end shear is $1880 \times 10 = 18,800$ pounds. With $v = 100$ lbs./in.^{2} and $j = 0.86$, as determined by subsequent calculation, the required area of stem is $b'd = 18,800/(100 \times 0.86) = 218$ square inches. The required depth is $218/12 = 18$ inches. This will provide sufficient room for the reinforcement, as shown in Fig. 32.

In making up the beam section, it is desirable to keep the section as shallow as practicable, in order to reduce the height of each story to a minimum. If the beams are made too shallow, the steel area must be large, and proper provision for negative moments at the supports is difficult. The dimensions assumed conform to good practice.

From the rules of the Joint Committee, the portion of the flange which is available as compression area is limited to $12t + b' = 12 \times 4 + 12 = 60$ inches, but not to exceed one-quarter of the span, which in this case is also 60 inches. The resulting tee-beam section is shown in Fig. 32 (b).

For a uniform load of 1880 pounds per foot, $M=1/12wl^2=1/12\times 1880\times 20\times 12=752{,}000$ in. lbs. positive moment at the span center and negative moment at the supports. At the span center the steel area required is $A=M/f_sjd=752{,}000/(16{,}000\times 0.92\times 18)=2.84$ square inches, where j is assumed as 0.92. To provide this area use seven $\frac{3}{4}$-inch round rods, or $7\times 0.442=3.09$ square inches. This area is somewhat in excess of that required, which is generally desirable in order to provide for negative moment conditions at the supports.

Checking back, as in Art. 152, it will be found that $f_s=14{,}700$ lbs./in.^{2}; $f_c=305$ lbs./in.^{2}; $k=0.235$, and $j=0.92$ as assumed.

The number of rods which must be run straight in order to provide for bond stress is determined by shear conditions at point B, Fig. 31. At this point the shear is $18{,}800-2\frac{1}{2}\times 1880=14{,}100$ pounds, and the circumference required for bond is $14{,}100/(120\times 0.92\times 18)=7.1$ inches. As the circumference of a $\frac{3}{4}$-inch rod is 2.36 inches, three rods must be run straight. This will leave four rods available for bending up as web reinforcement. The points of cut-off for these bars, which will be bent up in pairs, are shown at e and f, Fig. 32(a), as determined from eq. (2), Art. 137, with $l=15$ feet.

Before proceeding with the design of the web reinforcement, it is best to determine the reinforcement at the supports. This is usually provided by arranging the rods bent up for web reinforcement in such a manner that they can be carried to the upper side of the beam and used as tension reinforcement at the supports. As stated above, four rods can be bent up. Since the same number is available from adjacent beams, we have eight rods in place, as shown in Fig. 32(c). Rods a and b come from the beam to the right of the column and those marked with primes come from the beam to the left. The arrangement of rods is clearly shown in Fig. 32.

In determining the stresses in steel and concrete at the supports, the method given in Art. 87 will be used. Considering the beam section as rectangular with eight $\frac{3}{4}$-inch rods on the tension side, as shown in Fig. 32(c), we have $p=8\times 0.442/(12\times 18)=1.64\%$, and $R=M/bd^2=752{,}000/(12\times 18^2)=194$. From Plate II, $f_c=940$ lbs./in.^{2} and $f_s=14{,}200$ lbs./in.^{2} Although $R=194$ is outside the limits of the diagram of Plate II, it is possible to obtain the corresponding values of f_s and f_c by dividing R by any convenient quantity, say two, and determining the resulting fibre stresses. These, when multiplied by two, or whatever the divisor may have been, will give the desired fibre stresses for R.

According to the rules of the Joint Committee, the working stresses in continuous beams at points adjacent to supports may be increased

15%. Therefore, allowable $f_c=650\times1.15=748$ lbs./in.^{2} The concrete fibre stress determined above must be reduced $(940-748)/940=20.4\%$. This reduction can be brought about by placing compressive reinforcement in the lower face of the beam.

Since three rods had to be continued along the lower face of the beam to point B, Fig. 31, in order to provide for bond stress at B, we can continue these rods past the supports, thus providing the desired compressive reinforcement. As three rods are available from each side of the support, we can count on six rods if necessary. Fig. 32(c) shows that these rods are 2 inches from the extreme fibre, or $d'/d=\frac{2}{18}=0.11$. From Plate IX, with $d'/d=0.1$, we find that 0.7% compressive reinforcement must be provided in order to obtain the desired reduction in concrete stress. It will be convenient to use two rods from each side, or a total of four, as shown by rods c and c' in Fig. 32(c). These rods provide $p'=4\times0.442/(12\times18)=0.82\%$ of compressive reinforcement. Plate IX shows that this will reduce the concrete stresses by 23% and the tensile steel stresses by $3\frac{1}{2}\%$. The final stresses are $f_c=723$ lbs./in.^{2} and $f_s=13{,}700$ lbs./in.^{2} Also from Fig. 22 page 94, $k=0.44$ and $j=0.86$.

Fig. 32(a) shows the arrangement of the web reinforcement, which is a combination of bent rods and stirrups. A similar design is given in Art. 152. Shear diagrams are drawn for the tee-beam section, Fig. 32(a), and for the rectangular section, Fig. 32(d). The shears were calculated for a combination of dead load and maximum positive live-load shears. In each case a straight line variation of shear was assumed for points between the ends of the sections. Where the two shear diagrams overlap, the reinforcement must be able to carry the greater shear.

Assuming the concrete to carry a shear of 40 lbs./in.^{2}, it will be found that web reinforcement is not necessary at points more than 6 feet 10 inches from the supports. It will be assumed that one-third of the shear not carried by the concrete is carried by bent rods, and that the remaining shear is taken by stirrups. In the case under consideration it was found best to depart somewhat from these assumptions, for reasons which will be brought out in the following discussion. The shear diagrams of Figs. (d) and (e) show the relative amounts of shear carried by the several elements under the adopted conditions. It will be noted that shear reinforcement has not been provided for that portion of the shear diagram inside the column or cross girder.

The calculations for the tee-beam section showed that four rods are available for web reinforcement. These rods are shown in Fig. 32(a) bent up in pairs denoted by a and b. A single rod is also shown near the support. This rod is one of the three required for bond stress at B,

Fig. 31. As only two of these rods were continued over the support and used as compressive reinforcement, the third became available for web reinforcement, and is so used; it is indicated as bar *d*.

In distributing these rods over the shear area, it must be kept in mind that they are also to be used as tension reinforcement from the support to the quarter-point, as shown by the negative moment curve of Fig. 31. Their bending points at the top of the beam are determined by the steel area required for negative moment. Considered only as shear reinforcement, the bars should be bent up near the end of the beam, preferably at points such that the horizontal spacing of rods does not exceed the depth of beam. If so arranged it will be found for the beam of Fig. 32 that tension steel at the top of the beam out to the quarter-point can be provided only by introducing extra rods or by continuing rods from adjacent beams out to the quarter-point. This requires very long bars, which is undesirable. Since the shear which bars *a* could be assumed to carry is not large and is distributed over a considerable length of beam, it seemed best in this case to carry all of the shear by stirrups, as shown in the shear diagram of Fig. 32(*e*). Bars *a* can then be bent up close to their bend-up point, shown at *e*, Fig. 32(*a*), and used to provide tension reinforcement at the top of the beam from the quarter-point to the supports. They are so shown in Fig. 32(*a*). Bars *b* and *d* were arranged to take the shear shown by the shaded areas in the shear diagram of Fig. 32(*d*). The unit stresses in the bars will be about 4000 lbs./in.^{2}

Stirrups arranged as shown in Fig. 32(*a*) were designed to take the remaining shears. These stirrups consist of a $\frac{3}{8}$-inch rod in a single loop with unit stresses not to exceed 10,000 lbs./in.^{2} As stated in Art. 232, where negative moments exist the stirrups should be looped around the upper or tension rods, and for positive moments they should be looped around the lower rods. Since there is a section from the $\frac{1}{8}$ to the $\frac{1}{4}$ points where the moment may be either positive or negative, depending upon the loading conditions, the stirrups out to a point 3 feet from the support will be looped around the upper bars, and the remaining stirrups will be looped around the lower rods.

In determining the points to bend down or cut-off negative reinforcement it was assumed that the negative moment curve of Fig. 31 varied uniformly from zero at the quarter-point to a maximum at the supports, which is sufficiently accurate and on the side of safety. The required lengths of rods are then proportional to the steel area. On this assumption bars a' which furnish $\frac{1}{4}$ of the steel area can be cut off at a point $3\frac{3}{4}$ feet from the support. They are continued about a foot beyond this point in order to provide proper bond. In the same way bars b' can be cut off at a point $2\frac{1}{2}$ feet from the support, for bars a' and b' provide

half the required area. Bars b, which can be bent down at a point $1\frac{1}{4}$ feet from the support, are continued further than necessary in order to become effective as shear carrying area, as shown by the shear diagram of Fig. 32(d).

On the lower side of the beam at the support the cut-off points for compressive reinforcement are determined by methods similar to those used for tensile reinforcement. Bars c' are shown cut off at a point 2.5 feet from the support. Beyond this point there are three bars in place, which is ample.

The bond stress for tensile reinforcement at the support is 63.5 lbs./in^2 and for the compressive reinforcement it is 37.5 lbs./in.^{2} At the quarter point the bond is carried by two tension rods and the shear is 54.7 lbs./in.^{2} The bond stress is then 139 lbs./in.^{2} This is high, but since the rods are continuous beyond this point, the bond stress is actually distributed over a greater length of rod than assumed in the calculation.

Design of Girders.—As shown in Fig. 30, the girders carry two beams which frame into the girders at the third points. The loads from the beams are brought to the girders as concentrated loads. Each load will be equal to the reactions due to two beams, one on each side of the girder, or $2\times18{,}800=37{,}600$ pounds. In addition to these loads there is also a uniform load due to the weight of that portion of the girder which is below the floor slab. For the adopted girder section, which is 28 inches deep and 19 inches wide, as shown in Fig. 34, the weight is 475 pounds per foot. The total end shear is then 42,350 pounds. The area required for end shear is determined by methods similar to those used for the beam.

Fig. 33 shows the variation in positive and negative moments for loads at the third points of a beam. The discussion in Art. 221 shows that these moments can be determined by reducing the loads to be carried to an equivalent uniform load. Positive moment at girder center and negative moment at the supports will be given by $M=1/12wl^2$ reduced by 10%, where w is the equivalent uniform load and l is the span.

For the case under consideration, Fig. 30 shows that the live and slab load is represented by three loads of 37,600 pounds each. The equivalent load per foot is $37{,}600\times3/20=5640$ pounds per foot. To this must be added the weight of the girder, which is given above as 475 lbs./ft. The total load is then 6115 lbs./ft., and the bending moment is $0.9\times\frac{1}{12}\times6115\times20^2\times12=2{,}201{,}000$ in.-lbs. With $d=25$ inches and $j=0.93$, the steel area required at the girder center is 5.93 square inches. which will be provided by ten $\frac{7}{8}$-inch round rods with an area of 6.01 square inches. Bond stress at B, Fig. 33 requires five rods, and therefore five rods are available for web reinforcement.

Fig. 34(*c*) shows the rectangular beam section at the supports. Five bent-up rods from girders each side of the column are continued across the support to provide the tensile reinforcement. Three of the five rods required for bond at point *B*, Fig. 33, are continued across the support from adjacent girders. The tensile steel stress is found to be 16,490 lbs./in.^{2} and the concrete stress is 735 lbs./in.^{2} These values are satisfactory.

The web reinforcement, which consists of bent rods and stirrups, is shown in Fig. 34(*a*). Figs. (*d*) and (*e*) show the shear diagrams. The shear is practically uniform from the support to the third point. Be-

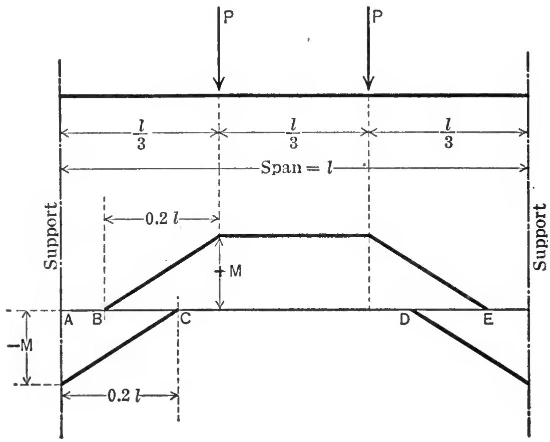


FIG. 33.

tween the third points the shear is small, and no web reinforcement is required.

As shown in Fig. 34(*a*) seven rods are available for web reinforcement. Five of these rods are provided by the tensile reinforcement at the center of the beam, and two rods, denoted by *d*, are rods required for bond at the end of the tee-beam section, but not required for compressive reinforcement at the support. With these rods it was possible to provide for one-third of the shear carried by the concrete, except for a short distance close to the third point, where it was necessary to use extra stirrups, as explained later. The shaded areas in the shear diagrams of Figs. 34(*d*) and (*e*) show the relative amounts of shear carried by the several rods. In this case the column is only slightly wider than the girder. Because of this the shaded area in Fig. 34(*d*) has been carried up to the face of the cross-beam. In calculating the bend-up and cut-off

points for both positive and negative moment reinforcement, the moment curves of Fig. 33 were assumed to be composed of straight lines. The cut-off points shown will be found to provide sufficient steel at all points.

The shear to be carried by stirrups varies from 40 lbs./in.^{2} at the supports to 32.8 lbs./in.^{2} at the third point. Stirrups consisting of $\frac{1}{2}$ inch round rods in a double loop will provide for this shear when spaced 10 inches apart. For a short distance to the left of the third point, where bent-up rods were not available, the shear to be carried is 49.1 lbs./in.^{2} Stirrups spaced 8 inches apart will carry this shear. In order to distribute the heavy loads brought by the cross-beam over a reasonable portion of the girder, a few stirrups were placed in the center portion of the girder. Fig. 34(a) shows the adopted stirrup spacing.

236. Columns.—In the general design of columns the recommendations of the Joint Committee (Appendix A) should be followed. Excessive quantities of steel and very high working stresses should be avoided. If large areas of steel (structural columns) are used, they should be self-sustaining and not rely upon the concrete for rigidity. Concrete may, however, be relied upon to transmit loads from girders to columns. Where small areas of steel are used the rods should be well lapped at the floor-level, and those from the lower columns should extend upwards the full depth of the connecting beams. The rods should be well banded together by steel bands or large wire so as to hold all parts in position and to strengthen the column circumferentially. Unless such banding is spaced very closely it should not be counted upon, however, as "hooping." Brackets under all connecting girders are serviceable in stiffening the frame as well as in decreasing the stress in the girders. Rods of connecting girders should pass well through the columns.

Large rods should be spliced by butt joints securely held together by sleeves. The splice is conveniently and properly made just above the floor level. Small rods may be spliced by lapping, care being taken to use sufficient lap to develop the necessary bond strength within the lap. At the base of columns large rods should butt against a base plate or be extended well into the footing, the controlling principle being to arrange the

end detail so as to develop the full assumed stress in the rod at the point where the column is of normal cross-section.

237. *Eccentric Loads on Columns.*—Where loads are applied on free brackets or cantilevers the load is definitely eccentric, and the moment due to the same can readily be calculated. This moment is resisted by the column, and if this continues above the bracket the resulting bending moments above and below the applied load are each approximately one-half the applied moment.

Moments are also caused in columns by unevenly loaded panels through the rigid beam connections. Assuming the beams rigidly fixed at the ends, a panel load on one without a load on the corresponding one on the opposite side will cause a bending moment in the beam at the column equal to $\frac{1}{12}pl^2$, where p = live load per lineal foot of beam. This moment is resisted mainly by the column and the members attached to it in the same plane as the loaded beam, and in proportion to their moments of inertia divided by their lengths. If the two beams are about as rigid as the column, then the moment in the column above and below the floor will be about one-fourth of the given moment, $=\frac{1}{48}pl^2$. This indicates, roughly, what may be expected from unequally loaded floors. In the lower stories of a high building such a moment would be of little consequence, but in the upper floors it might add a large percentage to the column stress.

238. **Footings.**—The problem of the design of footings is in general the same as that of floors. On account of the heavy concentrated loads and the large unit upward pressures of the earth against the footings the beam construction will be relatively heavy. The beams will be short and deep and will require special attention to provide against excessive shearing and bond stresses. For single footings of ordinary size a single symmetrical slab is most convenient. For larger footings and for footings carrying more than one column, a combination of beam and slab, similar to floor construction, is often most economical.

In the case of cantilever beams, such as footings, the maxi-

mum shearing stress occurs near the column base, where the moment is also a maximum. Diagonal cracks therefore tend to form first on the bottom and to slope inwards and upwards.

Bent rods, if used, must be bent up just outside the column, and not at the end of the beam, and stirrups must be spaced closely at this point. The beam being short it may require special attention to bond stress.

For large individual footings a beam and slab may be economical. To secure the benefit of a T-section and to give a flat upper surface the beam may be placed under the slab as shown in Fig. 35. This arrangement requires some atten-

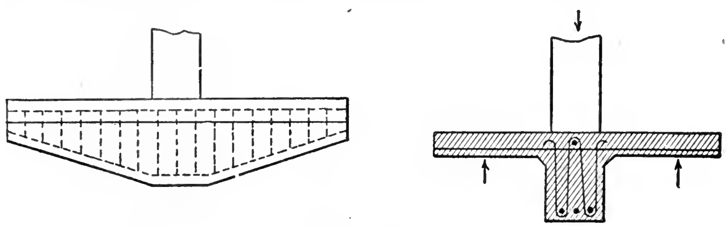


FIG. 35.

tion as to connection of slab to beam, as the upward pressure against the slab tends to pull it away from the beam. The use of an extra horizontal rod in the top of the main beam bonded by stirrups will give a thoroughly good anchorage for the transverse rods of the slab. For still larger areas a system of girders and beams may be adopted constituting a floor reversed as to loads.

239. Walls and Partitions.—In reinforced concrete buildings the framework is generally made self-sustaining and the walls supported at each floor level on beams or girders as in the usual steel frame building. Such walls (called curtain walls) may be made of brick, concrete, tile, or other suitable material. If of concrete they are usually 4 to 6 inches in thickness and reinforced both ways with $\frac{1}{4}$ to $\frac{1}{2}$-inch rods spaced $1\frac{1}{2}$ to 2 feet apart, chiefly for the purpose of preventing cracks or of localizing them to certain desired lines or contraction joints. Rods should be placed near the edges of openings. Where lateral

pressures occur, the beam stresses must of course be considered, but these generally are small. For partitions, tile or metal lath and plaster are generally to be preferred to solid concrete as being lighter and easier to install.

DURABILITY OF CONCRETE.

240. Fireproofing Effect of Concrete.—Severe fire tests show that when concrete is subjected to red-hot temperatures (about 1700°) for three or four hours and then is quenched by hose streams, it is likely to show pitting, but that it will still offer a sufficient protection to the steel.*

A reinforced-concrete building at Bayonne, N. J., was subjected to a very hot fire in the burning up of its contents but with no injury to the building.†

In the Baltimore fire of 1904 the value of concrete as a fireproofing material, and of reinforced-concrete construction, was fully demonstrated. Professor C. L. Norton of the Insurance Engineering Experiment Station, after a careful study of the damage done by the fire, states as follows.‡

"Where concrete floor arches and concrete-steel construction receive the full force of the fire it appears to have stood well, distinctly better than the terra-cotta." The reason for this he considers to be the fact that terra-cotta expands about twice as much as steel, but that concrete expands about the same. Little difference was observed between stone and cinder concrete. High temperatures long continued dehydrate and soften concrete, but this process in itself gives off water and absorbs the heat, thus protecting the interior. The layer of changed material is then a better non-conductor than before, so the process goes on very slowly. Captain J. S. Sewell, reporting to the Chief of Engineers § on the Baltimore fire, states

* See tests by Professor Ira W. Winslow in *Eng. Record*, Nov. 26, 1904, p. 634, and by Professor F. P. McKibben in *Eng. News*, Nov. 21, 1901, p. 378.

† *Eng. Record*, April 12, 1902, p. 341.

‡ *Eng. News*, June 2, 1904, p. 524.

§ *Eng. News*, March 24, 1904.

that, with reference to concrete construction subjected to very high heats: "Exposed corners of columns and girders were cracked and spalled, showing a tendency to round off to a curve of about 3 inches in radius. Where the heat was most intense the concrete was calcined to a depth of $\frac{1}{4}$ to $\frac{3}{4}$ inch, but showed no tendency to spall, except at exposed corners. On wide, flat surfaces the calcined material was not more than $\frac{1}{4}$-inch thick and showed no disposition to come off. The terra-cotta fireproofing showed up much poorer." In his general conclusions he considers it at least as desirable as steel work protected by the best commercial hollow tiles, and preferable to tile for floor slabs and fireproof covering.

One of the severest fire tests for reinforced-concrete buildings occurred December 9, 1914 at the Edison plant, West Orange, New Jersey. Several reinforced concrete buildings were subjected to exceedingly high temperatures on account of the inflammable nature of the material stored in them, and also by reason of the heat from the burning of adjacent structures. On account of these very severe conditions the concrete was considerably damaged, some of the heavily loaded parts of the structures collapsing. Much of the damage was caused by the use of fire streams on the hot concrete. The column reinforcement consisted of four vertical rods without ties or hooping. Except where the fire was most severe, the spalling on the columns extended to a depth of about $1\frac{1}{2}$ inches. Longitudinal cracks also occurred in many of the columns. Both trap rock and gravel were used as aggregates, with no apparent difference in the behavior of these two materials. Although the damage appeared at first to be very great, it was found practicable to repair the concrete framework of all the buildings at a cost of about 10% of the original construction. In executing this work defective material was cut away and new concrete of 1 : 3 mix was poured around the sound portions and held in place by means of wire cloth and additional reinforcement. One of the buildings in which the contents were entirely destroyed, but which contained no specially inflammable material,

was repaired at a cost of 2% of the original construction. In this fire all types of construction other than reinforced concrete, which included one sheet iron building and one brick building with steel frame, were completely wrecked.*

The necessary thickness of concrete to furnish adequate fire protection depends somewhat upon the character and importance of the member. Such members as main girders, where a failure would involve a considerable portion of the building and where the steel is concentrated in a few rods, should be more thoroughly protected than floor slabs of small span, where a few local failures would be of no importance, and where additional covering would add largely to the expense. Results of fire tests and experience in conflagrations indicate that 2 to 2½ inches will offer practically complete protection, and that a minimum of ½ to ¾ inch for floor slabs will usually be sufficient. Large flat surfaces, such as floor slabs, are less exposed than the corners of projecting forms like beams and columns.

241. The Protection of Steel from Corrosion.—A continuous coating of Portland cement has been found by experience to be a practically perfect protection of steel against corrosion. The rusting of iron requires the presence of moisture and carbon dioxide. Portland cement not only forms a coating which excludes the moisture and CO_2, but in hardening it absorbs CO_2, tending to remove any of this gas which may be present. In practice the protective nature of Portland-cement concrete has long been known, and its use as a paint was adopted by the Boston Subway Engineers after careful investigation.

While an unbroken coating of cement offers what appears to be a perfect protection, the value of a concrete as actually deposited may be very much less. A series of experiments made by Professor Charles L. Norton gives valuable information on this subject. In one series, small specimens of steel 6 inches long were embedded in blocks 3 by 3 by 8 inches in size of

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* *Eng. Record*, December 19, 1914, p. 654; and April 17, 1915, pp. 477 and 503.

various mixtures of cement, sand, and stone or cinders. The blocks were then exposed for three weeks to various corrosive atmospheres consisting of steam, air, and CO_2. The results were as follows: The neat cement furnished perfect protection. The specimens embedded in mortars and concretes showed spots of rust at voids or adjacent to a badly rusted cinder. He concludes that concrete to be an effective protection should be mixed quite wet so as to furnish a thin coating on the metal, and must be free from voids and cracks. He finds that dense cinder concrete mixed wet is as effective as stone concrete.

In a second series of experiments on steel already rusted, from a slight stain to a deep scale, the following results were obtained: The concrete was $1 : 2\frac{1}{2} : 5$ (stone) and $1 : 3 : 6$ (cinders). After one to three months in corroders and one to nine months in damp air no specimen showed any change except where the concrete was poorly applied. Some of the concrete was purposely made very dry and the rods were not well covered. These specimens were seriously corroded. Unprotected steel specimens subjected to the same treatment were almost entirely corroded. While the experiments of Professor Norton provided for a covering of $1\frac{1}{2}$ inches, there is no reason to suppose that a much thinner covering, if intact, will not furnish as good protection.

Many cases have been cited of steel removed from concrete after the lapse of 20 years or more and found to be in perfect condition. A test by Mr. H. C. Turner,* in which steel bars embedded to a depth of 3 inches in blocks of $1 : 2 : 4$ and $1 : 3 : 5$ concrete and exposed to sea-water and air for nine months showed perfect preservation.

Perfect protection of the steel by concrete was demonstrated in the case of a building at New Brighton, N. Y., built in 1902 and partly torn down in 1908. All steel was found to be in perfect condition excepting in a few cases where column hoops came closer than $\frac{3}{4}$ inch to the surface. The

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* *Eng. News*, Aug. 1904, p. 153.

footings were covered by the tide twice daily but the bars therein showed no corrosion.*

In view of such tests and observations as here noted it may be concluded that when well placed the concrete affords complete protection against corrosion.

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* *Eng. Record,* Vol. 57, 1908, p. 105.

CHAPTER X.

ARCHES.

242. Advantages of the Reinforced Arch.—If the loads on an arch were all fixed loads, it would be possible in any case to construct an arch ring so that the resultant pressure at all sections would intersect the centre of gravity of the section. The compressive stress at any section would then be uniformly distributed over the section, and the arch would be proportioned only for this uniform compression. The "line of pressure" would lie at the axis of the arch throughout. If, however, the arch ring is not made to fit the "line of pressure", or if part of the load is a live load, then the resultant pressure will not in general coincide with the axis of the arch. There will exist both bending and direct compression. If the resultant pressure and its position are known, the analysis of the stresses at any section is made in accordance with the method explained in Arts. 88–95, Chapter III.

In ordinary masonry or concrete arches tensile stresses are not permissible. The ring must therefore be designed so that the line of pressure will not pass outside the middle third. In reinforced arches this limitation does not exist. The arch rib is a beam, and if properly reinforced it may carry heavy bending moments involving tensile stresses in the steel.

Theoretically the gain in economy by the use of steel in a concrete arch is not great. If the pressure line does not depart from the middle third, the steel reinforces only in compression and in this respect is not as economical as concrete. If the line of pressure deviates farther from the centre, resulting in tensile stresses in the steel, the conditions are such that

those stresses must be provided for by the use of the steel at very low working values. That is to say, the direct compression in the arch is so large a factor that the limiting stresses in the concrete will always result in very small unit tensile stresses in the steel where any tension exists at all.

Practically the value of reinforcement is very considerable. It renders an arch a much more secure and reliable structure, it greatly aids in preventing cracks due to any slight settlement, and by furnishing a form of construction of greater reliability makes possible the use of working stresses in the concrete considerably higher than is usual in plain masonry. Furthermore, in long-span arches where the dead load constitutes by far the larger part of the load, any possible increase in average working stress counts greatly towards economy. It affects not only the arch but the abutments and foundations.

243. **Methods of Reinforcement.**—The reinforcement of arches is arranged in various ways. Since the arch is a beam subject to either positive or negative bending moments it is essential that it should be reinforced on both sides, but the shearing stresses due to beam action are relatively small, so that little is needed in the way of web reinforcement. The arch is also subjected to heavy compression, so that it is desirable that the inner and outer reinforcement be tied together, somewhat as in a column, although in this case the necessity therefor is much less.

A large proportion of the arches which have been constructed have been built according to some one of the various "systems" that have been devised. The most important of these systems are the Monier and the Melan. In the Monier system, invented about 1865, the reinforcement consists of wire netting, one net being placed near the intrados and one near the extrados. The longitudinal wires are made smaller than those following the arch ring, as they serve only to aid in equalizing the load and in preventing cracks. A large number of bridges have been built in Europe on this system.

In the Melan type, invented about 1890, the steel is in

the form of ribs of rolled I sections, or of built-up lattice girders, which are spaced two to three feet apart. The flanges constitute the principal reinforcement, but the web enables the steel frame to be self-supporting and to carry shearing stresses, and in the open lattice type it furnishes a good bond with the concrete. The Melan arch has been built extensively in this country, largely under the direction of Mr. Edwin Thacher.

Many arches are now being constructed in which reinforcing bars of any satisfactory form are employed without reference to any particular system, being used in accordance with the requirements of the case. The problem of reinforcement is quite as simple as in a beam, after the moments and thrusts in the arch have been found.

ANALYSIS OF THE ARCH.

244. General Method of Procedure.—The method of analysis presented here is based on the elastic theory and is of general application to arches of variable moment of inertia and loaded in any manner. It is mainly an algebraic method, although certain simple graphical aids may be used advantageously. It necessarily assumes that a preliminary design has been made by the aid of approximate or empirical rules or by reference to the proportions of existing arches. This arch is then exactly analyzed and the results used in correcting the design; the corrected design may then in turn be analyzed if it departs too greatly from the one first assumed. A discussion of the various rules for thickness of crown and form of arch will not be entered upon here. For this information the reader is referred to the various treatises on the arch, and especially to those of Professor Cain and Professor M. A. Howe. The work of Professor Howe on "Symmetrical Masonry Arches" * contains a very useful table of data of existing masonry and reinforced-concrete arches.

* New York, 1906. See also article by J. P. Schwada, Eng. News, Nov. 9, 1916.

The analysis of an arch consists in the determination of the forces acting at any section, usually expressed as the *thrust*, the *shear* and the *bending moment*, at such section. The thrust is here taken to be the component of the resultant parallel to the arch axis at the given point, and the shear is the component at right angles to such axis. The thrust causes simple compressive stresses; the shear causes stresses similar to those produced by the vertical shear in a simple beam.

The method of procedure will be to determine, first, the thrust, shear, and bending moment at the crown. These being known, the values of similar quantities for any other section can readily be determined. A length of arch of one unit will be considered.

245. Thrust, Shear, and Moment at the Crown (H_0, V_0, M_0).

Notation. (See Fig. 2.)

Let H_0 = thrust at the crown;

V_0 = shear at the crown;

M_0 = bending moment at the crown, assumed as positive when causing compression in the upper fibres;

N, V, and M = thrust, shear, and moment at any other section;

R = resultant pressure at any section = resultant of N and V;

δs = length of a division of the arch ring measured along the arch axis;

n = number of divisions in one-half of the arch;

I = moment of inertia of any section = $I_{\text{concrete}} + nI_{\text{steel}}$ (see Art. 90);

P = any load on the arch;

x, y = co-ordinates of any point on the arch axis referred to the crown as origin, and all to be considered as positive in sign;

m = bending moment at any point in the cantilever, Fig. 2, due to external loads.

Let AB, Fig. 1, be a symmetrical arch loaded in any manner with loads P_1, P_2, etc. Divide the arch into an even number of divisions (ten to twenty usually), making the divisions of such a length that the ratio $\delta s:I$ will be constant. This may be done by trial or by the more direct method explained in Art. 249. Mark the centre point of each division and num-

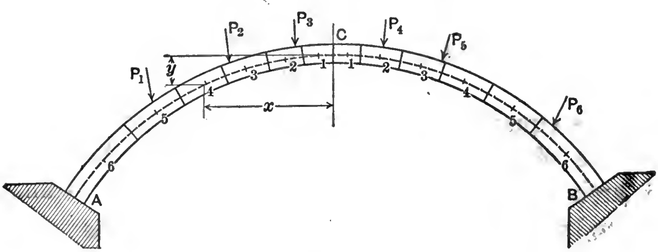

FIG. 1.

ber the points as shown. Consider the arch to be cut at the crown and each half to act as a cantilever subjected to exactly the same forces as exist in the arch itself, that is, the given loads, the reactions, and the stresses at the crown, represented

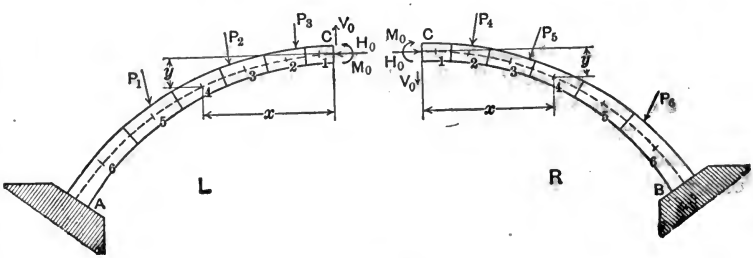

FIG. 2.

by H_0, V_0, and M_0 (Fig. 2). H_0 is applied at the gravity axis.

Let m = bending moment at any point 1, 2, 3, etc., due to the given external loads P (considering the arch as two cantilevers). Denote by m_R the moments in the right half and by m_L those in the left half of the arch. All the moments m

will be negative. The values of H_0, V_0, and M_0 will then be given by the following equations:

$$H_0=\frac{n\Sigma my-\Sigma m\Sigma y}{2[(\Sigma y)^2-n\Sigma y^2]}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$V_0=\frac{\Sigma(m_R-m_L)x}{2\Sigma x^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$M_0=-\frac{\Sigma m+2H_0\Sigma y}{2n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In these equations the summations Σy, Σy^2, and Σx^2 are for one-half of the arch only; the summation Σm is for the entire arch and is equal to $\Sigma m_R+\Sigma m_L$; the summation $\Sigma(m_R-m_L)x$ is a summation of the products $(m_R-m_L)x$, in which m_R and m_L are the bending moments at corresponding points in the right and left halves which have equal abscissas x; and the summation Σmy is for the entire arch, but since symmetrical points have equal y's this quantity may be calculated as $\Sigma(m_R+m_L)y$. A positive result for V_0 indicates action as shown in Fig. 2. All quantities are readily calculated. Distances should be scaled and quantities tabulated.*

* *Demonstration.*—Consider the left-hand cantilever of Fig. 2. Under the forces acting the point C will deflect and the tangent to the axis at this point will change direction (the abutment at A being fixed). Let Δy, Δx, and $\Delta\phi$ be, respectively, the vertical and horizontal components of this motion and the change in angle of the tangent. Then according to the principles relating to curved beams [1] we have the values

$$\Delta y=\Sigma Mx\frac{\delta s}{EI}, \quad \Delta x=\Sigma My\frac{\delta s}{EI}, \quad \text{and} \quad \Delta\phi=\Sigma M\frac{\delta s}{EI}, \quad . \quad . \quad . \quad (a)$$

in which the various quantities have the same significance as in Art. 245.)

In like manner, referring to the right cantilever, let $\Delta y'$, $\Delta x'$, and $\Delta\phi'$ represent the components of the movement of C and the change of angle of the tangent. These may be expressed in terms similar to Eq. (a).

Now evidently

$$\Delta y=\Delta y', \quad \Delta x=-\Delta x', \quad \text{and} \quad \Delta\phi=-\Delta\phi'. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Furthermore, since $\delta s/I$ is constant and likewise E, the quantity $\delta s/EI$ may be placed outside the summation sign.

[1] See Church's Mechanics, or Johnson's Framed Structures, p. 236.

246. Thrust, Shear, and Moment at any Section.—The values of H_0, V_0, and M_0 having been found, the total bending moment at any section, 1, 2, etc., is

$$M=m+M_0+H_0y\pm V_0x. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The plus sign is to be used for the left half and the minus sign for the right half of the arch.

The resultant pressure, R, at any section of the arch is equal in magnitude to the combined resultant of the external loads between the crown and the section in question, and the forces H_0 and V_0. These resultants can best be found graphically. The thrust, N, is the component of the resultant, R, parallel to the arch axis and the shear, V, is the component perpendicular to this axis.

247. Partial Graphical Calculation.—Where the loads are vertical the calculation of the quantities m can be advantageously performed by means of an equilibrium polygon as follows:

Let AB, Fig. 3, represent the arch axis. The load line is

Using the subscript L to denote left side and R to denote right side we then derive the relations

$$\left.\begin{aligned}\Sigma M_Lx&=\ \ \Sigma M_Rx,\\ \Sigma M_Ly&=-\Sigma M_Ry,\\ \Sigma M_L\ &=-\Sigma M_R.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

The moment M may in general be expressed in terms of known and unknown quantities thus:

$$M_L=m_L+M_0+H_0y+V_0x \text{ for the left side}$$

and

$$M_R=m_R+M_0+H_0y-V_0x \text{ for the right side.}$$

Hence, substituting in (c) and combining terms, and noting that ΣM_0 for one half is equal to nM_0, we have

$$\Sigma m_Lx-\Sigma m_Rx+2V_0\Sigma x^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

$$\Sigma m_Ly+\Sigma m_Ry+2M_0\Sigma y+2H_0\Sigma y^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

$$\Sigma m_L+\Sigma m_R+2nM_0+2H_0\Sigma y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

From Eq. (d) is obtained Eq. (2), Art. 245; and from Eqs. (e) and (f) are obtained Eqs. (1) and (3), noting that $\Sigma m_L+\Sigma m_R=\Sigma m$, and $\Sigma m_Ly+\Sigma m_Ry=\Sigma my$.

a-c-b. Select any convenient pole O on a horizontal line through the point c, at the junction of loads P_3 and P_4, the loads adjacent to the crown C. Construct the equilibrium polygon $efgh$, producing to i and k the segment fg between loads P_3 and P_4. Drop verticals from the points 1, 2, 3, etc. The desired bending moments m, at the several points, will then be equal to the corresponding intercepts z_2, z_3, etc., on these verticals between the equilibrium polygon and the line ik, multiplied by the pole distance H; or in general $m = Hz$.

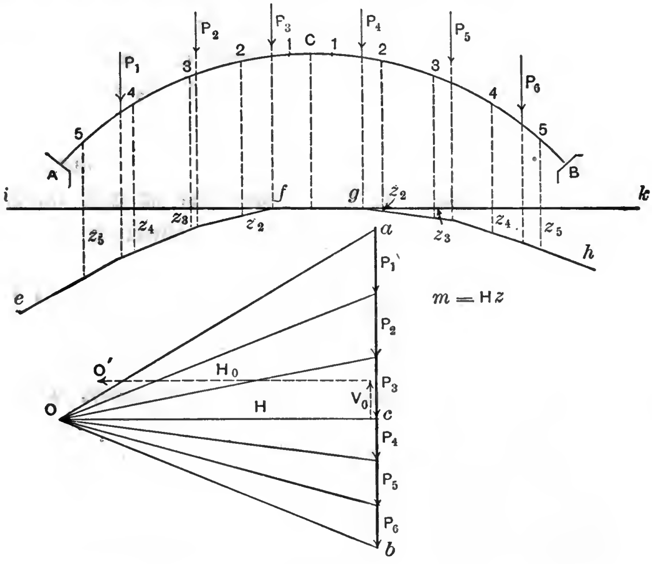

FIG. 3.

Finally, after the values of H_0, V_0, and M_0 are found by Eqs. (1), (2), and (3), the true equilibrium polygon can be drawn, if desired, and values of thrust, shear, and moment at various points determined graphically. The true pole is found by laying off V_0 and H_0 from the point c. The distance of the equilibrium polygon above or below the axis at the crown is equal to M_0/H_0. It lies above the axis if the result is positive and below if negative. The equilibrium

polygon is then drawn from the crown each way towards the ends.

Where the loads are inclined at various angles it is still possible to use a graphical process for getting values of m, but there is little to be gained in such a case. After the values of H_0, V_0, and M_0 are found, however, it will be expedient to draw a final equilibrium polygon, or line of pressure, as explained above.

248. General Observations.—The method of analysis just described is brief, general, and easily followed. The arithmetical calculations are not longer than those required in the usual graphical process, while the graphical aids here suggested are of the simplest character.

The loads and their points of application have been considered apart from the divisions of the arch ring, as the two things are in no wise related. Where no spandrel arches are used and the entire load is applied continuously along the arch ring, the load may for convenience be divided to correspond with the arch divisions and applied at the center points, 1, 2, 3, etc. This division is, however, of no importance, the only requirement being a sufficiently small subdivision of the arch ring and of the load so that the errors of approximation will be negligible. Where spandrel arches are used, the live load and a large part of the dead load will be applied at the centers of the arch piers. The weight of the main arch ring may also be considered as concentrated at these same points.

If calculations are to be made for more than one loading it will be noted that the denominators of the values for H_0, V_0, and M_0 do not change. The quantities involving m are the only ones requiring recalculation, and if the load on but one-half of the arch is changed, then the values of m for that half only need be recalculated. In the case of a symmetrical loading, or a load on one-half only, the calculation of m is also necessary for one-half the arch only. For symmetrical loads, $V_0=0$.

249. Division of Arch Ring to give Constant $\delta s/I$.—In most cases the depth of the arch ring increases from crown towards springing line, giving a variable moment of inertia. Considering the concrete only, the moment of inertia will increase as d^3 so that a comparatively small change in depth will cause a large change in moment of inertia. To maintain $\delta s/I$ constant, the value of δs will therefore be much greater near the springing line than at the crown, and hence to secure the desired accuracy the length of division at the crown will need to be made fairly short. The value of $\delta s/I$ to adopt so that there will be no fractional division may be determined as follows:

Let $i=\frac{1}{I}$;

i_a = mean value of i;

s = half length of the arch ring measured along the axis;

n = desired number of divisions in one-half the arch.

Calculate first the mean value of i for the half arch ring by determining several values at equal intervals along the arch. Then the desired value of $\delta s/I$ is

$$\frac{\delta s}{I}=\frac{s i_a}{n} = \frac{s}{n I_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In determining the value of I the steel reinforcement must be duly considered.

The value of $\delta s/I$ being known, the proper length of δs for any part of the arch ring can readily be determined. Beginning at the crown, the length of the first division is determined, then the second, third, etc., to the end. The length of a division not being exactly known beforehand, the value of I for that division will not be exactly known, but the necessary adjustment is very simple.

A graphical method of doing this is shown in Fig. 4. A curve EF is drawn whose ordinates are the values of I and whose abscissas are distances along the arch axis measured from the crown. Beginning at one end, preferably the springing line, a series of similar isosceles triangles is drawn between the base and the curve EF, having a ratio of base to altitude equal

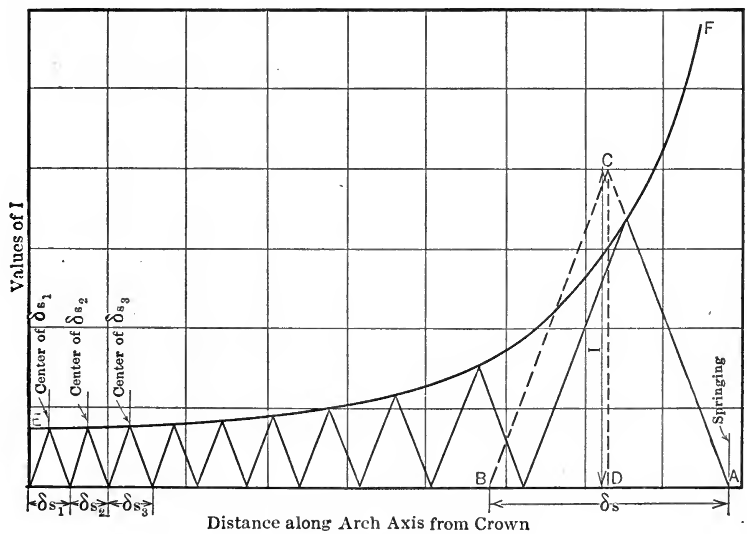

FIG. 4.

to the known value $\delta s/I$. This is conveniently done by first drawing a construction triangle ABC with any assumed value of I for altitude DC and the corresponding value of δs for base AB. The several triangles will have sides parallel to ACB. The construction having been completed and adjusted if necessary the correct lengths of the several divisions and the location of their center points can be taken from the diagram.

250. Temperature Stresses.—For a rise of temperature of t degrees the increase in span-length, if the arch be not restrained, would be equal to ctl, where c = coefficient of expan-

sion and l=span. The restraint of the abutments induces a thrust H_0 at the crown given by the equation

$$H_0=\frac{EI}{\delta s}\cdot\frac{ctln}{2[n\Sigma y^2-(\Sigma y)^2]}.^* \quad \ldots \ldots \quad (6)$$

The summations refer to one-half the arch.

Having determined H_0, then

$$M_0=-\frac{H_0\Sigma y}{n}.^* \quad \ldots \ldots \quad (7)$$

The bending moment at any point is

$$M=M_0+H_0y. \quad \ldots \ldots \quad (8)$$

The thrust and shear at any point in the arch are found by resolving H_0 parallel and normal to the arch axis at that point. Graphically, the true equilibrium polygon is a horizontal line drawn a distance below the crown equal to $M_0/H_0=\Sigma y/n$.

251. Stresses Due to Shortening of Arch from Thrust.—A thrust throughout the arch producing an average stress on the concrete equal to f_c lbs/in² would shorten the arch span an amount equal to f_cl/E if unrestrained. This action develops horizontal reactions in the same manner as a *lowering* of tem-

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* *Demonstration.*—For temperature stresses $\Delta\phi$ of Eq. (a), Art. 245, is zero and Δx is equal to the change in length of the half-span, $=\frac{ctl}{2}$. We therefore have

$$\Sigma M_Ly\frac{\delta s}{EI}=\frac{ctl}{2},$$

and

$$\Sigma M_L=0.$$

In this case, there being no external loads, $m=0$, and from symmetry, $V_0=0$, hence $M=M_0+H_0y$. Substituting this value of M in the above equations we have

$$M_0\Sigma y+H_0\Sigma y^2=\frac{ctl}{2}\cdot\frac{EI}{\delta s},$$

and

$$nM_0+H_0\Sigma y=0.$$

From these are readily derived Eqs. (6) and (7).

perature. The value of the resulting reactions, or the crown thrust, may then be found by substituting $f_c l/E$ for ctl of Eq. (6). There results

$$H_0 = -\frac{I}{\delta s} \cdot \frac{f_c l n}{2[n\Sigma y^2 - (\Sigma y)^2]}. \quad . \quad . \quad . \quad . \quad (9)$$

The moments at crown and elsewhere are given by Eqs. (7) and (8), using the value of H_0 from Eq. (9).

The thrusts and moments due to arch shortening will not usually be large. They may be applied as corrections to the thrusts and moments found before.

252. Deflection of the Crown.—The downward deflection of the crown under a load is given by Eq. (a), Art. 245. It is

$$\varDelta y = -\frac{\delta s}{EI} \Sigma Mx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If M is not determined for all points, use the value of M from Eq. (4), deriving

$$\varDelta y = -\frac{\delta s}{EI}[\Sigma mx + M_0 \Sigma x + H_0 \Sigma xy + V_0 \Sigma x^2]. \quad . \quad . \quad (11)$$

The summations are for one-half only.

The *rise* of crown due to an increase of temperature is obtained from Eq. (11) by substituting from Eqs. (6) and (7). There results

$$\varDelta y = \frac{ctl}{2} \cdot \frac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma y^2 - (\Sigma y)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

253. Unsymmetrical Arches.—If the arch is unsymmetrical the value of $\delta s/I$ should be made constant for the entire arch, and the number of divisions made even as before. The central point of the arch with reference to the *number* of divisions may then be taken as the crown, and the X-axis made tangent to the arch at this point. The two halves of the arch are now unlike and all terms resulting from substitution in Eq. (c), p. 349, must be retained. Explicit formulas for H_0,

V_0, and M_0 are very cumbersome, but the three equations derived from (c) are as follows:

$$(\Sigma_L x - \Sigma_R x)M_0 + (\Sigma_L xy - \Sigma_R xy)H_0 + \Sigma x^2 V_0 = \Sigma_R mx - \Sigma_L mx, \quad (13)$$

$$\Sigma y M_0 + \Sigma y^2 H_0 + (\Sigma_L xy - \Sigma_R xy)V_0 = -\Sigma my, \quad . \quad . \quad (14)$$

$$2nM_0 + H_0 \Sigma y + (\Sigma_L x - \Sigma_R x)V_0 = -\Sigma m. \quad . \quad . \quad (15)$$

Where no subscript is used the summation is for the entire arch. Numerical values of the coefficients of M_0, H_0, and V_0 should be obtained and the equations then solved.

254. Application of the Preceding Theory.—*Example 1.*—An arch ring will be assumed of the dimensions shown in Fig. 5. Span length with reference to the axis = 30 ft., rise = 8 ft. Thickness at crown = 1 ft., at springing lines = 1 ft. 6 in. For a small arch such as this great accuracy is not needed, hence a small number of divisions may be used. The number will be four for each half. These divisions are determined so that $\delta s/I$ is constant. The loads are applied at the centre-points 1, 2, 3, 4, and are assumed to be somewhat inclined (excepting loads P_4 and P_5), the several vertical and horizontal components being as given in the figure.

TABLE A.

CALCULATIONS FOR H_0, V_0, AND M_0.

1	2	3	4	5	6	7	8	9
Point.	x	y	x^2	y^2	m_L	m_R	$(m_L+m_R)y$	$(m_R-m_L)x$
1	1.55	.09	2.40	.01	0	0	0	0
2	4.90	.68	24.01	.46	− 13,840	− 8,640	− 15,300	+ 25,500
3	8.45	2.10	71.40	4.41	− 46,800	− 29,560	− 160,400	+ 145,700
4	12.85	5.35	165.12	28.62	−116,680	− 75,490	−1,028,100	+ 529,300
Σ		8.22	262.93	33.50	−291,010		−1,203,800	+700,500
Springing.	15.00	8.00			−180,820	−120,640		

Eq. (1) gives $H_0 = \dfrac{4(-1203800)-(-291010\times 8.22)}{2[(8.22)^2-4\times 33.50]} = +\ 18{,}230$ lbs.

Eq. (2) gives $V_0 = \dfrac{700500}{2\times 262.93} = +\ 1{,}330$ lbs.

Eq. (3) gives $M_0 = -\dfrac{-291010 + 2\times 18230\times 8.22}{2\times 4} = -1{,}090$ ft-lbs.

TABLE B.

BENDING MOMENTS, THRUSTS, AND ECCENTRIC DISTANCES.

1	2	3	4	5	6	7	8	9
Point.	H_0y	V_0x	Bending Moment M.		Thrusts.		Eccentric Distances.	
			Left.	Right.	Left.	Right.	Left.	Right.
1	1,640	2,060	+2,610	−1,520	18,450	18,640	+.14	−.08
2	12,400	6,530	+4,000	−3,860	19,580	19,310	+.21	−.20
3	38,300	11,260	+1,650	−3,620	22,050	20,770	+.07	−.17
4	97,500	17,120	−3,100	+3,850	28,800	24,970	−.11	+.15
Springing.	145,900	20,000	−16,070	+4,140	28,800	24,970	−.56	+.17

The calculations of the several quantities in the formulas for H_0, V_0, and M_0 (Art. 245) are given in Table A. The co-ordinates x, y of the several points are given in cols. 2 and 3; then x^2 and y^2 in cols. 4 and 5; then in cols. 6 and 7 are given the quantities m_L and m_R, considering each half-arch a cantilever. These are readily calculated. Thus, on the left, for point 1, $m=0$; for point 2, $m=4130\times(4.90-1.55)=13,840$; for point 3, $m=4130\times(8.45-1.55)+5035\times(8.45-4.90)+310\times(2.10-.68)=46,800$; and for point 4, $m=4130\times(12.85-1.55)+5035\times(12.85-4.90)+5950\times(12.85-8.45)+310\times(5.35-.68)+725(5.35-2.10)=116,680$. The value of m at the springing line is also calculated and placed in this table for future use. The moments on the right are similarly found. All moments m are negative. In cols. 8 and 9 are then given the products $(m_L+m_R)y$ and $(m_R-m_L)x$.

Substituting in Eqs. (1), (2), and (3), Art. 245, there are obtained the values for H_0, V_0, and M_0 given below the table.

The values of the bending moments, thrusts, and shears at any point may now be found either graphically or algebraically. The force-diagram method will be much the better for obtaining thrusts and shears; the moments may then be obtained either

by constructing an equilibrium polygon or by the application of Eq. (4), p. 349.

In Fig. 5, the graphical construction is given. The load-line is a-c-b. The true pole is found by laying off $V_0 = +1330$ from point c (at the junction of the loads adjacent to the crown, P_4 and P_5); then $H_0 = 18{,}230$ horizontally to O. The

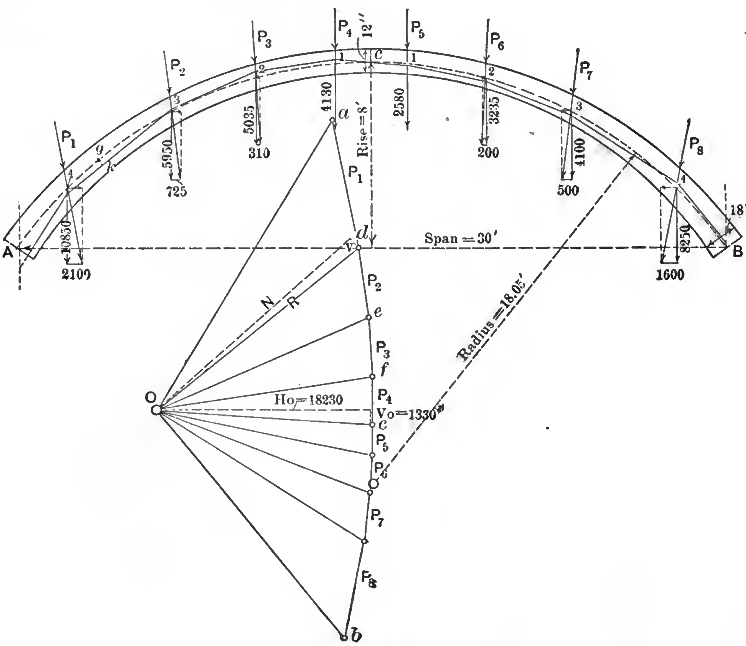


FIG. 5.

force diagram is drawn and then the equilibrium polygon, beginning at the crown and drawing the segment 1–1 at a distance *below* the crown equal to $1090/18230 = .06$ ft. The resultant, R, acting at any section may be scaled from the force polygon, and the moment at any point will be equal to this resultant multiplied by the perpendicular distance at that point from the arch axis to the equilibrium polygon. For example, the bending moment at point g is equal to the force Od multiplied by the arm gk. The tangential component of the

resultant R (the true thrust N) may be found by resolving the force R parallel and normal to the arch axis at the point in question. In most cases the thrust N may be taken as equal to R. The shear V will be the normal component of R; it will not usually require consideration.

Table B contains calculated values of moments and eccentric distances for points 1, 2, 3, 4 and the springing lines. The moments are calculated from the formula (Eq. (4)) $M = m + M_0 + H_0 y \pm V_0 x$. The quantities m are obtained from Table A, cols. 6 and 7. The thrusts are scaled from the force polygon, being in each case the thrust on the abutment side of the point in question. The eccentric distances are equal to the moments divided by the thrusts; they are of use in calculating stresses in the arch. Obviously the bending moment at any other point, such as g, may be calculated in the same way as those here given.

255. *Example 2.* (Fig. 6).—For another example an arch will be assumed of 100-ft. span and 20-ft. rise; thickness at crown $= 30$ in., thickness at springing line $= 42$ in. It will be assumed that the roadway is supported on spandrel piers 10 ft. apart, thus concentrating most of the load at points 10 ft. apart as shown; the weight of the arch ring will also be assumed as applied at these points. The loads as given represent an arch with live load on the left half only. The half arch is divided into ten divisions, making $\delta s/I$ constant. The loads in this case are vertical, so that the graphical method may be used to advantage in determining the cantilever moments m. The load line a-c-b is drawn and a pole O' selected on a horizontal line through c at the center of the crown load P_5. The pole distance is H. An equilibrium polygon, $efgh$, is then drawn, and the moments m will be equal to the intercepts, z, from this polygon to the horizontal line ik, multiplied by H. These moments, and the remainder of the calculations for H_0, V_0, and M_0 are given in Table C. The true pole O is then found as before and the correct equilibrium polygon drawn. The thrusts are then scaled from the force polygon, and the eccentric

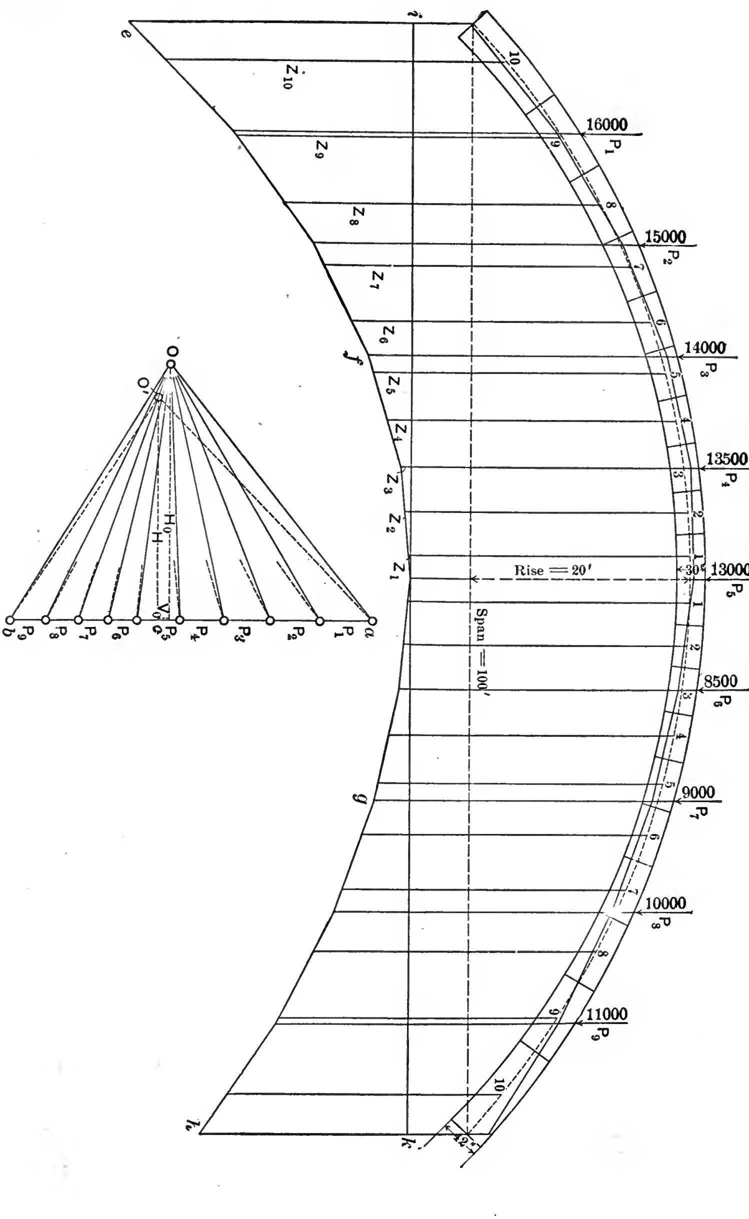

16000
15000
14000
13500
13000
8500
9000
10000
11000
Rise = 20′
Span = 100′
30°
12″

FIG. 6.

distances from the equilibrium polygon. These are given in Table D, together with resulting bending moments. The bending moments may also be calculated as done in Example 1.

Table C.

CALCULATIONS FOR H_0, V_0, AND M_0.

1	2	3	4	5	6	7	8	9
Point	x	y	x^2	y^2	m_L	m_R	$(m_L+m_R)y$	$(m_R-m_L)x$
1	2.07	.07	4.3	.00	− 13,500	− 13,500	− 2,000	——
2	5.96	.30	35.5	.09	− 38,700	− 38,700	− 23,000	——
3	10.00	.72	100.0	.52	− 65,000	− 65,000	− 94,000	
4	14.18	1.40	201.1	1.96	− 148,600	− 127,700	− 387,000	+ 296,000
5	18.43	2.37	339.7	5.62	− 233,600	− 191,400	− 1,007,000	+ 777,000
6	23.06	3.77	531.8	14.21	− 369,000	− 288,400	− 2,479,000	+ 1,859 000
7	28.06	5.64	787.4	31.81	− 539,000	− 408,400	− 5,348,000	+ 3,665,000
8	33.60	8.23	1129.0	67.73	− 781,400	− 577,400	− 11,183,000	+ 6,854,000
9	39.60	11.80	1568.2	139.24	− 1,075,400	− 781,400	− 21,910,000	+ 11,642.000
10	46.40	16.80	2153.0	282.24	− 1,511,000	− 1,083,000	− 43,580,000	+ 19,859,000
Σ		51.10	6850.0	543.42	$\underbrace{\qquad}$ −8,350,100		− 86,013,000	+ 44,952,000

$$H_0=\frac{10\times(-86013000)-(-8350100)\times 51.10}{2.[(51.10)^2-10\times 543.42]}=+76,760 \text{ lbs.},$$

$$V_0=\frac{44952000}{2\times 6850}=+3,280 \text{ lbs.}$$

$$M_0=-\frac{-8350100+2\times 76760\times 51.10}{2\times 10}=+25,260 \text{ ft-lbs.}$$

Table D.

THRUSTS, ECCENTRIC DISTANCES, AND MOMENTS.

1	2	3	4	5	6	7
Point.	Thrusts.		Eccentric Distances.		Bending Moments.	
	Left.	Right	Left.	Right.	Left.	Right.
1	76,800	77,400	+ .31	+ .13	+ 23,700	+ 10,100
2	76,800	77,400	+ .38	− .13	+ 29,200	− 10,100
3	78,600	78,800	+ .61	− .22	+ 48,000	− 17,300
4	78,600	78,800	+ .39	− .53	+ 30,700	− 41,700
5	78,600	78,800	+ .44	− .57	+ 34,700	− 45,000
6	82,700	81,500	+ .26	− .61	+ 21,200	− 49,700
7	82,700	81,500	+ .14	− .52	+ 11,400	− 42,400
8	89,300	85,300	− .15	− .36	− 13,400	− 30,700
9	89,300	85,300	− .16	+ .23	− 14,300	+ 19,600
10	98,600	90,700	− .44	+ .88	− 43,400	+ 80,000
Springing	98,600	90,700	− .21	+ 1.67	− 20,700	+ 152,000

Temperature Stresses.—Suppose in Ex. 2 it is desired to know the thrust and bending moment at the crown due to a rise of temperature of 30°. Eqs. (6) and (7), Art. 250, will be used. Assume $E=2,000,000$ lbs/in² $=288,000,000$ lbs/ft². Suppose the value $\delta s/I$, in foot-units, is 3.1. Then from Eq. (6)

$$H_0=\frac{288,000,000}{3.1}\times\frac{.000006\times30\times100\times10}{2(10\times543-(51.1)^2)}=2970 \text{ lbs.},$$

$$M_0=-2970\times\frac{51.1}{10}=-15,200 \text{ ft-lbs.}$$

The equilibrium polygon is a horizontal line drawn a distance below the crown equal to $15200/2970=5.11$ ft. The moment at any point is equal to the thrust H_0 multiplied by the vertical distance from such point to this equilibrium polygon. At the springing line, $M=H_0\times(20-5.11)=2970\times14.89=44,200$ ft-lbs. This may also be calculated by Eq. (8).

Stresses Due to Shortening of Arch.—The modification of the thrust due to the compressive deformations of the arch ring is found by Eq. (9). The average compressive stress at any section is found by dividing the thrust at that section by the area of the transformed section of arch ring. This is nearly uniform throughout the arch and equal to about 150 lbs/in². Then

$$H_0=-\frac{1}{3.1}\times\frac{150\times144\times100\times10}{2[10\times543-(51.1)^2]}=1240 \text{ lbs.}$$

This thrust is equal to 42% of the thrust due to temperature change, already found. The resulting moments and stresses will then be 42% of those due to temperature change. They will be of opposite sign.

256. Maximum Stresses in the Arch Ring.—From the values of thrust, moment, and eccentric distance as given in Tables B and D, the stresses in the concrete and steel can be found at any section of the arch, as explained in Chap. III, Arts. 88–95. The maximum value of fibre stress

will be where the sum of the stresses due to thrust, N, and moment, M, is a maximum. This will not in general be where either the thrust or the moment is a maximum; but as the thrust varies slowly along the arch ring the maximum stress will occur very near to the point of maximum moment.

The position of live load causing maximum moment at any point will differ in arches of different proportions. In designing an arch it is sufficient generally to determine the maximum stresses at the crown, the haunch, and the springing line. This will require several different positions of the live load. For the crown the maximum positive moments are caused when a short length of the arch (one-fourth to one-third) at the center is loaded, and the maximum negative moments when the remaining portions are loaded. The maximum positive and negative moments at the haunch (about the $\frac{1}{4}$ point) are caused when the arch is loaded about two-thirds the span length and one-third the span length respectively. The same loading will give practically the maximum moments at the springing lines.

These conditions make it desirable to analyze the arch for various assumed loadings about as follows: full load; one-third of span loaded; two-thirds of span loaded; center third loaded; and end thirds loaded. In the case of large and important structures it may be found desirable to place the loads somewhat differently than here indicated. A complete and exact solution can readily be made by analyzing the arch for a load of unity at each load-point of one-half of the arch. Influence lines can then be drawn for moment or fibre stress and the exact maximum values determined.

257. Example of Complete Analysis for Maximum Stresses.—To further illustrate the methods of analysis here described, and the use of influence lines in determining maximum stresses, a complete analysis will be made of the arch of Fig. 6, modified, for sake of illustration, by the addition of steel reinforcement amounting to 2 in.^{2} per foot of arch along both the extrados and the intrados, and placed 3 in. from the surface.

Calculation of Values of I and of δs.—The half length of the arch axis is found to be 55.17 ft. The depth at crown = 2.5 ft., and at springing line = 3.5 ft. The moment of inertia at any section $= I = I_c + 15I_s$, where I_c and I_s = moment of inertia of the concrete and steel sections respectively. Following the procedure of Art. 249, the half arch will first be divided into a convenient number of equal divisions and the value of I determined at the center of each division. The reciprocal, i, is then found. It is convenient to use the same number of preliminary divisions as is desired for the final divisions. The results of these calculations are given in Table E. The calculations are made in foot-units. The first part of the table relates to the preliminary ten equal divisions, each = 55.17/10 = 5.517 ft. long. The value of $\delta s/I$ is by Eq. (5), Art. 249, equal to $\frac{55.17 \times .363}{10} = 2.00$. Then in Fig. 7 (*a*), the several

Table E.

DIVISIONS OF ARCH RING.

Properties of Preliminary Equal Divisions.						Properties of Final Divisions.		
No. of Division.	Depth, d	I_c	$15I_s$	$I = I_c + 15I_s$	$i = \frac{1}{I}$	δs	I	d
1	2.55	1.38	.44	1.82	.549	3.6	1.8	2.53
2	2.65	1.55	.48	2.03	.492	3.8	1.9	2.59
3	2.75	1.73	.53	2.26	.442	4.2	2.1	2.67
4	2.85	1.93	.57	2.50	.400	4.6	2.3	2.75
5	2.95	2.14	.62	2.76	.362	5.0	2.5	2.84
6	3.05	2.36	.68	3.04	.328	5.4	2.7	2.93
7	3.15	2.61	.73	3.34	.300	6.0	3.0	3.03
8	3.25	2.86	.79	3.65	.274	6.7	3.35	3.15
9	3.35	3.13	.85	3.98	.251	7.5	3.75	3.28
10	3.45	3.42	.91	4.33	.231	8.4	4.2	3.42
					3.629	55.2		

$i_a = 3.63/10 = .363$ By Eq. (5) $\delta s/I = \frac{55.17 \times .363}{10} = 2.00.$

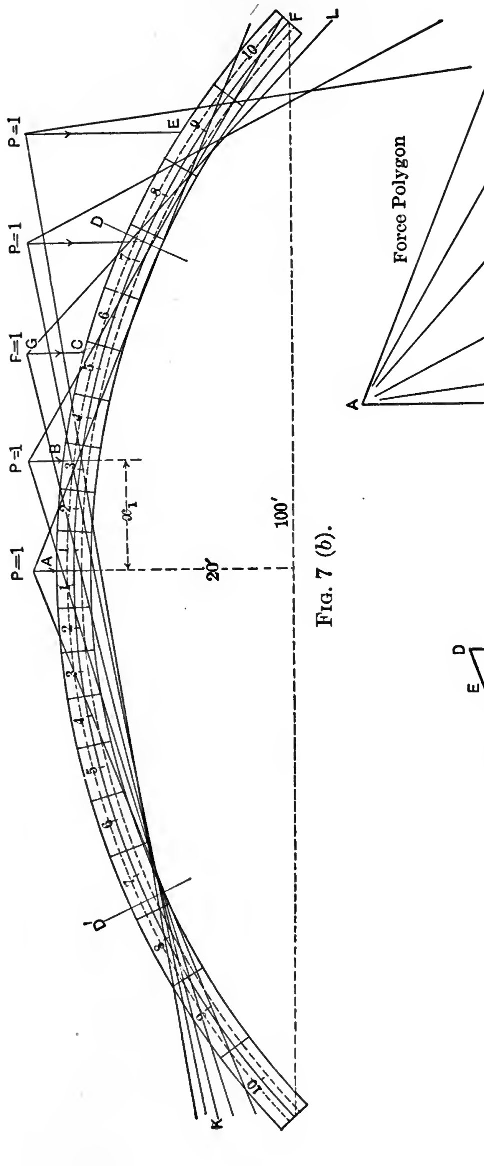


FIG. 7 (b).

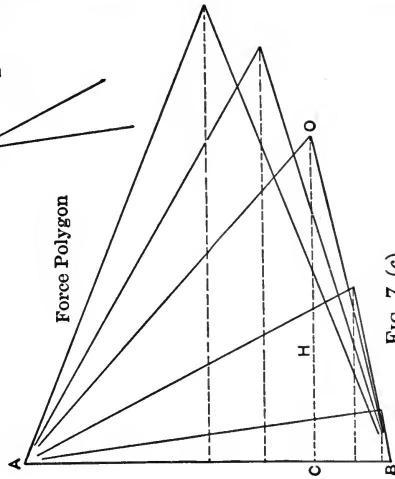


FIG. 7 (c).

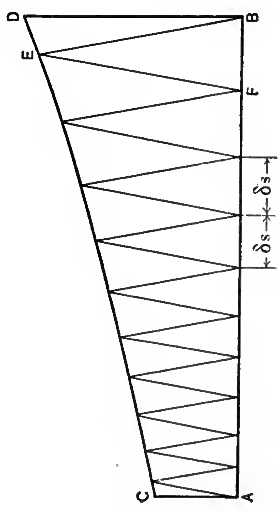


FIG. 7 (a).

values of δs are found by the graphical method described in Art. 249. From the values of I given in Table E the curve CD is plotted, the length of the base AB being 55.17 ft. Beginning then at B, the similar isosceles triangles BEF, etc., are drawn, each having a ratio of base to altitude ($\delta s/I$) equal to 2.00, according to the scale used. The several values of δs are then scaled off or may be found (more accurately) by multiplying the scaled values of I at the apexes of the triangles by 2.00. The resulting values of δs are given in the latter part of Table E, where are also given the values of I and δ for the center points of the final subdivisions.

Calculation of H_0, V_0, and M_0 for a Load Unity at Each Load Point.—Fig 7 (*b*) shows the arch ring with the subdivisions made and the center points numbered 1, 2, 3, etc., as in Fig. 6. The load points considered are those on the right, A B, C, D, and E. These are the center points of the supporting spandrel walls. [In the case of an arch with continuous loading, arbitrary load points may be selected sufficiently close to secure the desired degree of accuracy in the influence lines. Four of five such points will generally be sufficient.] The problem now is, first, to calculate H_0, V_0, and M_0, for a load unity placed successively at these several load points. The computations are performed in the same manner as in the preceding examples. Table F gives all the calculations. The quantities m are equal to $x-x_1$, where x_1 is the abscissa of the load point in question, measured from the crown. For the load point A the entire load is, for convenience, assumed as acting just to the right of the crown and hence as belonging to the right half. The shear V_0 is evidently equal to 0.5 in this case. The quantities m_L do not appear. For load points B, C, D, and E the calculations become progressively less numerous.

From Table F we then have for the denominators of eqs. (1), (2), and (3) of Art. 245,

$$2[(\Sigma y)^2-n\Sigma y^2]=2[(52.54)^2-10\times 578.62]=-6051.4,$$

$$2\Sigma x^2=2\times 7026=14{,}052, \qquad 2n=20.$$

TABLE F.

CALCULATIONS FOR H_0, V_0, AND M_0.

Points.	x	y	x^2	y^2	Load at A; $x_1=0$.		
					m_R	m_Ry	m_Rx
1	1.78	0.02	3.2	.00	1.78	.0	3.2
2	5.49	0.29	30.1	.08	5.49	.8	30.1
3	9.47	0.62	89.7	.27	9.47	2.9	89.7
4	13.76	1.32	189.3	1.74	13.76	9.1	189.3
5	18.38	2.37	337.7	5.61	18.38	21.7	337.7
6	23.34	3.86	544.7	14.89	23.34	45.0	544.7
7	28.67	5.91	821.8	34.92	28.67	84.7	821.8
8	34.37	8.67	1181.5	75.10	34.37	148.9	1181.5
9	40.44	12.33	1635.5	151.93	40.44	249.3	1635.5
10	46.82	17.15	2192.5	294.08	46.82	401.5	2192.5
Σ	....	52.54	7026.0	578.62	−222.52	−1927.9	−7026.0

TABLE F—(*Continued*).

Points.	Load at B; $x_1=10.0$.			Load at C; $x_1=20.0$.		
	m_R	m_Ry	m_Rx	m_R	m_Ry	m_Rx
4	3.76	4.9	51.7			
5	8.38	19.8	154.1			
6	13.34	51.5	311.2	3.34	12.9	78.6
7	18.67	110.3	535.1	8.67	51.2	248.4
8	24.37	211.2	837.6	14.37	124.6	494.6
9	30.44	375.2	1231.3	20.44	252.0	826.6
10	36.82	631.5	1724.3	26.83	460.0	1255.2
Σ	−135.78	−1404.4	−4845.3	−73.65	−900.7	−2903.4

Points.	Load at D; $x_1=30.0$.			Load at E; $x_1=40.0$.		
	m_R	m_Ry	m_Rx	m_R	m_Ry	m_Rx
8	4.37	37.9	150.4			
9	10.44	128.7	422.5	0.44	5.4	17.8
10	16.83	288.5	787.9	6.83	117.0	31.96
Σ	−31.64	−455.1	−1360.8	− 7.27	−122.4	−337.4

The values of H_0, V_0, and M_0 then result as follows:

$$\text{Load at } A \begin{cases} H_0=\dfrac{10\times(-1927.9)-(-222.52)\times 52.54}{-6051.4}=+1.254; \\ V_0=\dfrac{-7206}{14,052}=-0.50; \\ M_0=-\dfrac{-222.52+2\times 1.254\times 52.54}{20}=+4.54. \end{cases}$$

$$\text{Load at } B \begin{cases} H_0=\dfrac{10\times(-1404.4)-(-135.78)\times 52.54}{-6051.4}=+1.142; \\ V_0=\dfrac{-4845.3}{14,052}=-0.345; \\ M_0=-\dfrac{-135.78+2\times 1.142\times 52.54}{20}=+0.80. \end{cases}$$

$$\text{Load at } C \begin{cases} H_0=\dfrac{10\times(-900.7)-(-73.65)\times 52.54}{-6051.4}=+0.849; \\ V_0=\dfrac{-2903.4}{14,052}=-0.207; \\ M_0=-\dfrac{-73.65+2\times 0.849\times 52.54}{20}=-0.78. \end{cases}$$

$$\text{Load at } D \begin{cases} H_0=\dfrac{10\times(-455.1)-(-31.64)\times 52.54}{-6051.4}=+0.477; \\ V_0=\dfrac{-1360.8}{14,052}=-0.097; \\ M_0=-\dfrac{-31.64+2\times 0.477\times 52.54}{20}=-0.92. \end{cases}$$

$$\text{Load at } E \begin{cases} H_0=\dfrac{10\times(-122.4)-(-7.27)\times 52.54}{-6051.4}=+0.139; \\ V_0=\dfrac{-337.4}{14,052}=-0.024; \\ M_0=-\dfrac{-7.27+2\times 0.139\times 52.54}{20}=-0.37. \end{cases}$$

Calculation of Moments and Thrusts at any Given Section of the Arch Due to Unit Loads at the Load Points.—Consider, for

example, a load unity at C, Fig. 7 (b). The values of H_0, V_0, and M_0 being known, the moments, shears, and thrusts at any given section can now be found either graphically or analytically, as already explained. The graphical method is much the simpler. For this purpose draw the force polygon, Fig. 7 (c), the load AB being equal to unity, and lay off the shear $V_0 = -.207$ from B, upwards to C, and then the thrust $H_0 = .849$ to the right, fixing the pole O. Then from the center of the arch at the crown measure down a distance equal to $e = \frac{M_0}{H_0} = \frac{.78}{.849} = .92$ ft., fixing one point on the equilibrium polygon. Lines KG and GL drawn parallel to OB and AO, intersecting in the load vertical at G, complete the polygon. The other polygons are drawn in a like manner.

Having these polygons drawn, the bending moment at any section, due to any one of the unit loads, is equal to the vertical ordinate measured from the proper equilibrium polygon to the gravity axis of the arch, multiplied by the corresponding pole distance. From these several polygons the moments can therefore be found at any section for a unit load at any load point on either half of the arch, and the influence line drawn for such moment. The tangential thrusts can likewise be determined.

Influence Lines for Fiber Stresses at any Section.—Instead of constructing influence lines for moment and thrust it will be more direct to construct them at once for stress on extreme fiber.

From Art. 92 the stress on extreme fiber is

$$f_c = \frac{Mu}{I} + \frac{N}{A},$$

in which u is the distance from the neutral axis to the fiber in question and N is the thrust normal to the section. The bending moment is equal to the thrust times its eccentricity, or $M = Ne$.

If r = radius of gyration of the section, we have $Ar^2 = I$. The above expression for f_c may then be written in the form,

$$f = \frac{Neu}{I} + \frac{N\frac{r^2}{u}\cdot u}{I}$$

$$= \frac{N\left(e+\frac{r^2}{u}\right)u}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The quantity $\frac{r^2}{u}$ is a length, and $N\left(e+\frac{r^2}{u}\right)$ is a moment which may be written M'. Then we have

$$f = \frac{M'u}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This new moment, M', is equal to the thrust N, multiplied by the eccentricity e plus the additional distance $\frac{r^2}{u}$. We may then compute the value of $\frac{r^2}{u}$ for the section considered and take the center of moments at this distance above or below the neutral axis, according as the lower or upper fiber stress is desired. By so selecting the center of moments the fibre stress becomes equal to the moment multiplied by the usual factor $\frac{u}{I}$, and therefore varies with the moment. An influence line drawn for such moment will therefore serve as an influence line for fiber stress.

In this problem influence lines have thus been drawn for upper and lower fiber stress at sections taken at the various load points and at the springing line F. They are given in Fig. 8. To explain further their construction consider the section at D, and suppose the arch is loaded with a unit load at C. The arch near D is shown to a larger scale in Fig. 9. The force R is the resultant of the forces on the

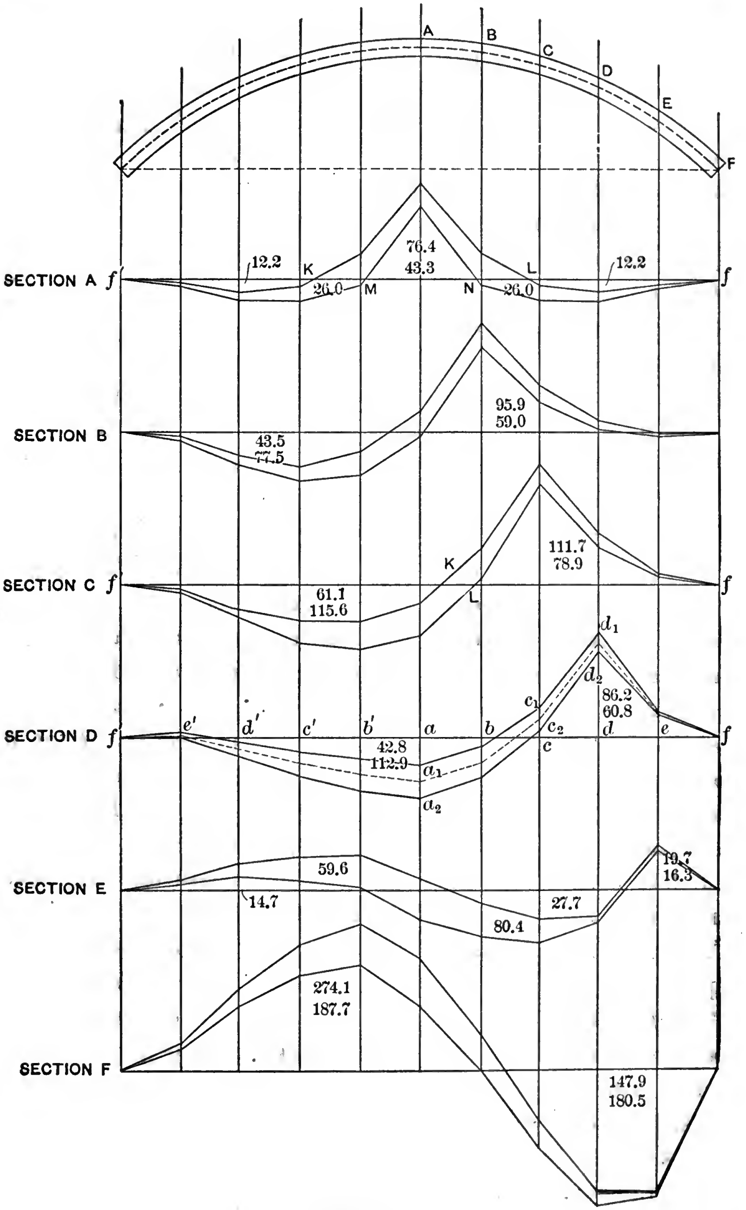

A
B
C
D
E
F
SECTION A
f
12.2
K
76.4
43.3
L
12.2
f
26.0
M
N
26.0
SECTION B
43.5
77.5
95.9
59.0
SECTION C
f
K
L
61.1
115.6
111.7
78.9
f
SECTION D
f
d_1
d_2
86.2
60.8
c_1
e'
d'
c'
b'
a
b
c_2
d
e
f
c
42.8
112.9
a_1
a_2
SECTION E
59.6
19.7
16.3
14.7
27.7
80.4
SECTION F
274.1
187.7
147.9
180.5

Fig. 8.

left and acts on the line GL, Fig. 7 (b). In amount it is equal to AO in the force polygon. The neutral axis is at a, Fig. 9, and the distances ab_1 and ab_2 are the values of r^2/u calculated for this section. The point b_1 is therefore the center of moments for M' for upper fiber stress and b_2 is the center of moment for M' for the lower fiber stress. (The points b_1 and b_2 are at the edges of the "kern" of the section. In a simple rectangular section they are at the edges of the middle third and in a truss or plate girder, where the flange takes all the moment, they are at the flange centers.)

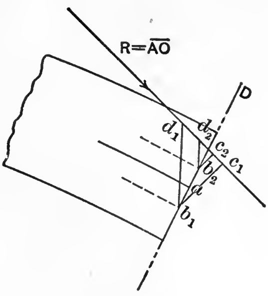


Fig. 9.

These values of M' are given graphically by $R \times b_1c_1$ and $R \times b_2c_2$, or by the more convenient products $H \times b_1d_1$ and $H \times b_2d_2$. These moments are calculated and plotted in Fig. 8 at point C, under the load unity. They are the ordinates cc_1 and cc_2, and are plotted as positive ordinates since they represent positive bending moments. In the same manner values of the moments are determined at section D for unit loads at A, B, C, and E, and plotted at corresponding points in the diagram. For loads on the left half of the structure the values are found by considering the section D to be on the left half of the arch at D', and measuring the several ordinates to the reaction lines as drawn for loads on the right. The complete influence lines are given in the diagram. The upper line $f'a_1d_1f$ is for upper fiber stress and the lower line $f'a_2d_2f$ is for lower fiber stress. The dotted line (midway between, for symmetrical sections) is the influence line for the usual bending moment, center of moments at the neutral axis. Since positive ordinates indicate positive moments it follows that for the upper fibers positive ordinates indicate compression and for the lower fibers positive ordinates indicate tension. Negative ordinates indicate respectively tension

on the upper fibers and compression on the lower. The actual fiber stress at section D due to a unit load at any point is now given by the ordinate to the influence line at the point, multiplied by $\frac{u}{I}$ for this section.

In Table G are given values of u, I, r^2, and $\frac{r^2}{u}$ for each of the sections $A \dots F$. The distances r^2/u are shown by the dotted lines in the arch ring, Fig. 7 (*b*).

TABLE G.

PROPERTIES OF SECTIONS $A \dots F$.

Section.	Depth, d.	$u=\frac{1}{2}d$.	I	A (A_c+15A_s)	r^2	$\frac{r^2}{u}$
A	2.50	1.25	1.70	2.92	.58	.46
B	2.78	1.39	2.10	3.20	.65	.47
C	3.00	1.50	2.54	3.42	.74	.49
D	3.18	1.59	3.06	3.60	.85	.53
E	3.35	1.67	3.70	3.77	.98	.59
F	3.50	1.75	4.46	3.92	1.14	.65

Where the arch is continuously loaded the influence lines should be drawn as smooth curves through the points determined by the ordinates at the load-points selected.

Maximum Fiber Stress at Any Section.—Having the influence lines drawn for several sections, as in Fig. 8, the maximum fiber stress at the various sections can readily be determined for any given loading, either uniform or concentrated. In the figure the various positive and negative areas above and below the axis have been measured and are written within the respective areas. The upper figure refers in all cases to the area between the axis and the upper curve. For uniform loads these areas, multiplied by the load per foot, give the respective moments. For concentrated loads the moments are found by summing the products of the several loads times the corresponding ordinates. The maximum values can readily be determined by trial. The influence lines show very clearly the extent and general position of loads for maximum stresses.

For example, at the crown, section A, the moment for upper

fiber stress, due to a uniform dead load of, say, 800 pounds per foot, $=[76.4-(2\times12.2)]\times800=41{,}600$ ft.-lbs. The stress $f_c=M'\frac{u}{I}=41{,}600\times\frac{1.25}{1.70}=20{,}700$ lbs/ft², $=330$ lbs/in². For a uniform live load of 500 lbs. per foot, the maximum stress is caused when the load extends from K to L, and the stress for such load $=76.4\times500\times\frac{1.25}{1.70}\div144=195$ lbs/in².

Again, at section C, the dead load upper fiber stress $=(111.7-61.1)\times800\times\frac{1.50}{2.54}\div144=166$ lbs/in²; and the maximum live load stress $=111.7\times500\times\frac{1.50}{2.54}\div144=230$ lbs/in², the load extending from K to f. For the maximum lower fiber stress the load extends from f' to L and the stress $=237$ lbs/in²; the dead load stress $=120$ lbs/in². Other stresses are found in a similar manner.

The influence lines show that in arches of proportions such as here considered, the loading for maximum stresses is about as noted in Art. 256. For the crown, the center third should be loaded; for the haunch (sections C and D) the load should extend over about two-thirds or one-third the span length, and about the same for the section at the springing line.

Case in which the Resultant Stress is Tensile.—If the resultant stress is tensile at any section and the tension in the concrete is to be neglected then the formulas for fiber stress of Art. 93 must be used, which makes it necessary to plot the influence lines for the true bending moment and to determine the thrust as well as the moment. The diagram, Plate XIII, can then be used.

258. Illustrative Examples of Arch Design.—Fig. 10 shows a longitudinal section and part plan of a bridge at Grand Rapids, Mich.* It consists of five spans of lengths from 79 to 87 feet. The reinforcement is composed of $1\frac{1}{4}$-in. Thacher bars spaced 14 in. apart near both the extrados and intrados. Each pair is connected by $\frac{5}{8}$-in. connecting-rods spaced 4 ft. apart.

* Eng. News, Vol. LII, 1904, p. 490.

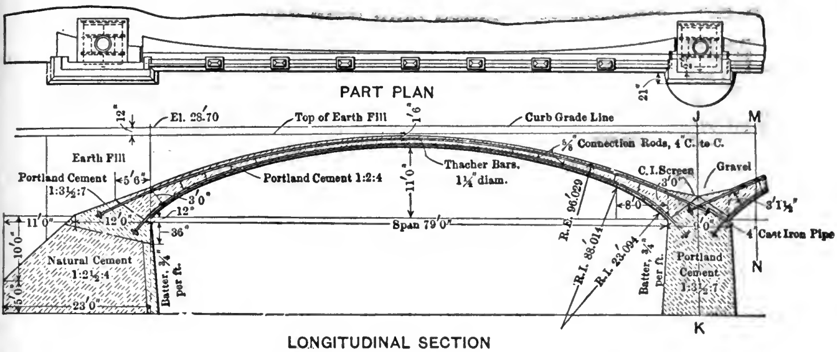


FIG. 10.—Arch Bridge at Grand Rapids, Mich.

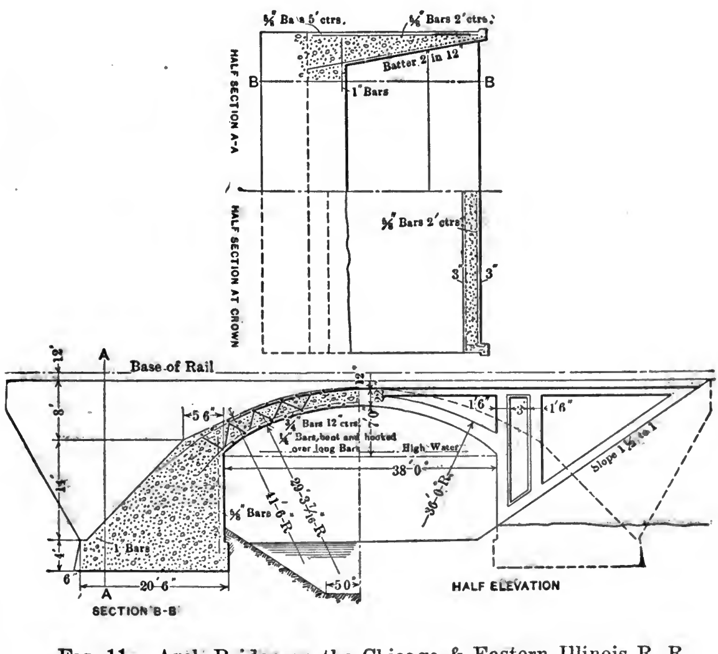


FIG. 11.—Arch Bridge on the Chicago & Eastern Illinois R. R.

Fig. 11 illustrates a 38-ft. span reinforced arch on the C. & E. I. R.R.* Johnson corrugated bars were used.

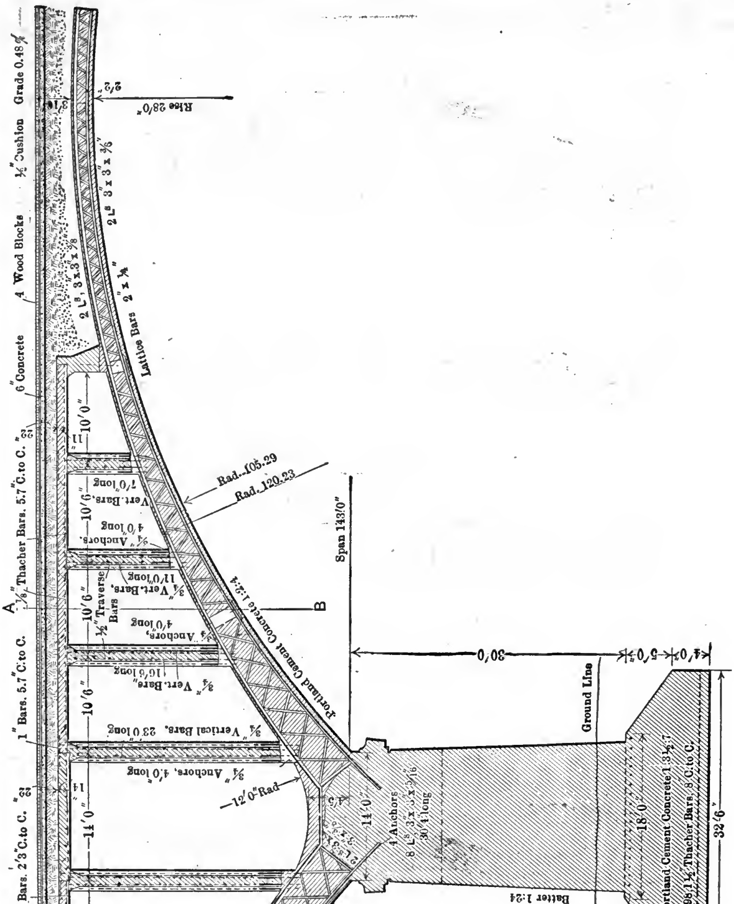


Fig. 12.

Figs. 12 and 13 show the details of a design for a concrete viaduct at Milwaukee, Wis.† The design was made by the

* R. R. Gaz., April 13, 1906, p. 390.
† Eng. News, Vol. LVII, 1907, p. 178.

Concrete-Steel Engineering Co., of New York City, and is a typical Melan arch. The reinforcement consists of built-up ribs of $3'' \times 3'' \times \frac{3}{8}''$ angles connected by lattice bars $2'' \times \frac{1}{4}''$.

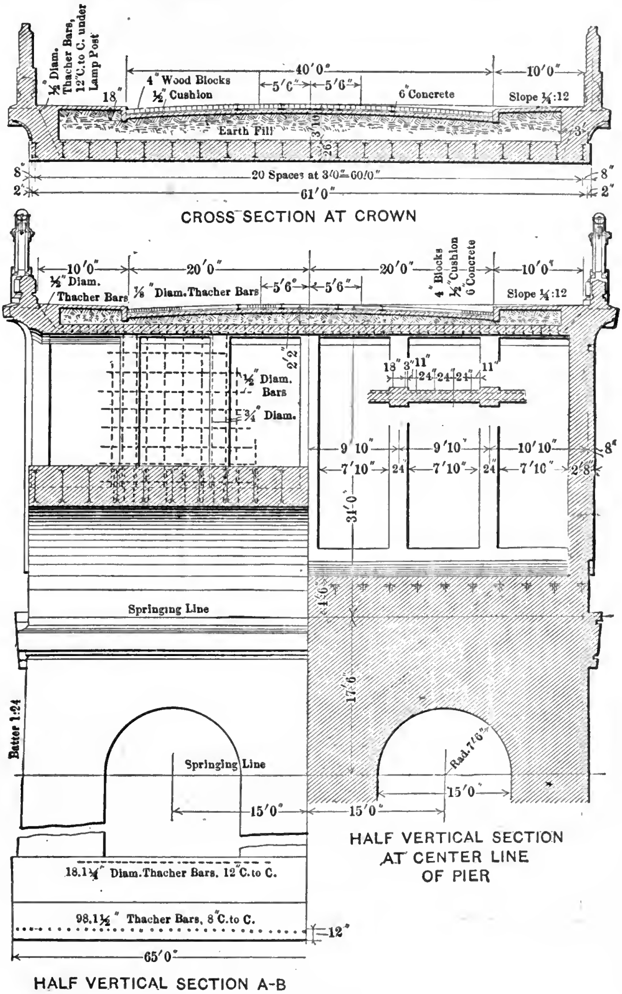


FIG. 13.

The ribs are spaced 3 ft. apart. These ribs are designed to carry the entire bending moment at a stress of 18,000 lbs/in². The stress in the concrete was limited to 500 lbs/in² in com-

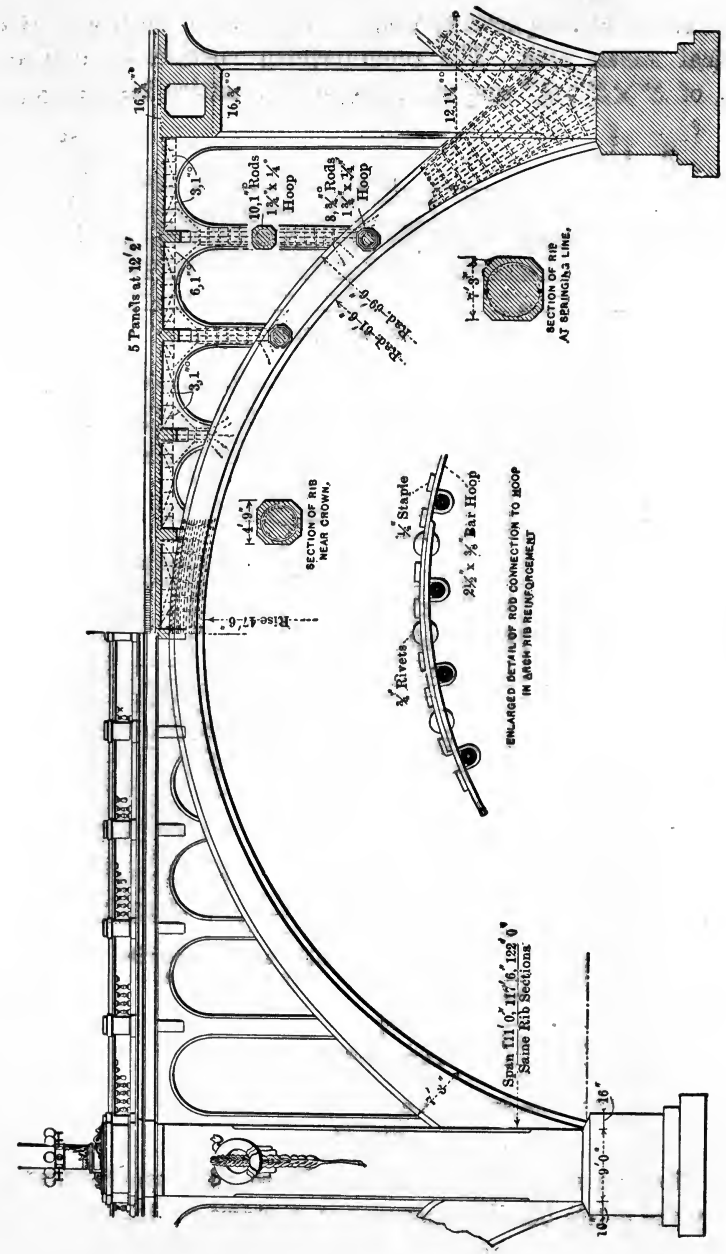

5 Panels at 12'2'
10,1"° Rods
1¾" x ¼" Hoop
8,¾"° Rods
1¾" x ¼" Hoop
SECTION OF RIB AT SPRINGING LINE.
SECTION OF RIB NEAR CROWN.
Rise 47'6"
¼" Staple
2½" x ⅜" Bar Hoop
¾" Rivets
ENLARGED DETAIL OF ROD CONNECTION TO HOOP IN ARCH RIB REINFORCEMENT
Span 111'0", 117'6", 122'0"
Same Rib Sections

FIG. 14.

pression, or 600 lbs/in^2 including temperature stresses. The roadway is supported over a considerable portion of the span length by means of a reinforced floor carried on vertical walls.

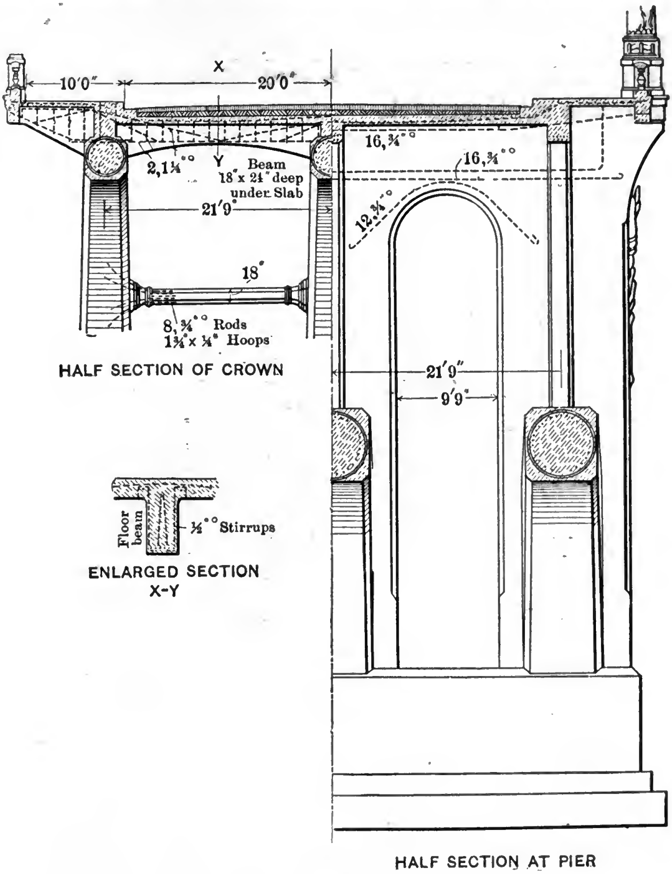


FIG. 15.

The viaduct contains eight spans of the dimensions shown in the illustration.

Figs. 14 and 15 illustrate another design for the same viaduct mentioned in the preceding paragraph.* This design was

* Eng. News, Vol. LVII, 1907, p. 178.

submitted by Mr. C. A. P. Turner, of Minneapolis, Minn. In this case the arch is composed of three ribs 4 ft. 9 in. square at the crown and 7 ft. 3 in. square at the pier. The rib reinforcement is composed of longitudinal rods arranged in a circle and connected at frequent intervals by bands of $2\frac{1}{8}'' \times \frac{3}{8}''$ metal. The stirrups and bent bars in the floor-beams and slabs give a very effective reinforcement.

CHAPTER XI.

RETAINING-WALLS AND DAMS.

259. Advantages of Reinforced Concrete.—Retaining-walls, dams, bridge abutments, and the like constitute a class of structures in which the outside forces acting are mainly horizontal, and in which, therefore, the question of stability is largely a question of safety against overturning. Where ordinary masonry is used in these structures the weight of the material must be depended upon to balance the overturning forces; for though the structure be anchored to the foundation no tensile stresses can be allowed in the masonry. As a consequence of these limitations the maximum compressive stresses in such structures are not high, except in extreme cases, so that generally the dimensions are determined by the weight of the material. The application of reinforced concrete in such cases enables the design to be so modified as to utilize the weight of the material to be retained as part of the resisting weight and to calculate the sections to develop more nearly the full strength of the concrete. A very considerable gain in economy therefore results.

RETAINING-WALLS.

260. Method of Determining Stability.—No attempt will be made here to present the various mathematical theories of earth pressure. Unless the results obtained from such theories are carefully controlled by the results of experience they are apt to be very misleading. Probably the most satisfactory way to design a reinforced concrete retaining-wall, as regards stability against overturning, is to proportion it so that it will be,

as nearly as possible, equivalent to a solid masonry wall of such a section as is known to have given satisfactory results under the given conditions. Rules of practice as to solid masonry walls have long been established. They represent the accumulated experience of many engineers and are based upon data obtained from many failures as well as from successful designs. Until experience is had directly with the reinforced type of wall its stability may, therefore, well be determined by comparison with the older form of construction. The analysis given here will consequently be limited to a convenient method of comparison of the two types. It may be said in passing that good construction requires quite as much attention to the earth filling itself and to its drainage as to the design and construction of the wall.

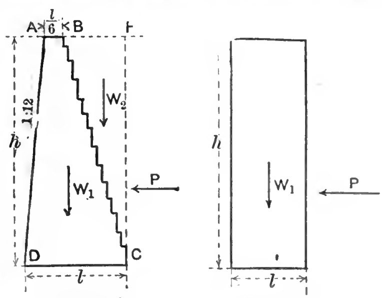
In dimensioning a reinforced concrete wall which will possess stability equal to that of a given solid wall, it will be convenient to determine the equivalent fluid pressure under which the solid wall will be stable and then apply this pressure to the reinforced type of wall. The basis of the calculation of this fluid pressure will be to determine the weight per cubic foot of a fluid which will exert such a pressure against the solid wall as to cause the resultant of all forces above the base to intersect the base at the edge of the middle third. If, then, the reinforced wall be designed so that it will be equally stable against this pressure, it will be practically equivalent to the solid wall.

It will be seen that this method is very simple and adapts itself readily to the utilization of present rules of practice. If desired, the theory of earth pressure may of course be directly applied to the problem.

261. Equivalent Fluid Pressure for Ordinary Masonry Walls.—Two forms of wall will be considered (Fig. 1). Form (*a*) is the more common form of wall. A small batter is usually given to the front face, and the back face is sloped in an irregular line, the width of the top being as narrow as circumstances may warrant. Such a wall will be stable when the width of the base is made from one-third to one-half the height,

four-tenths being a common rule of practice. Form (b) is used for relatively low walls. Its width may be a little less than that of form (a) for equal stability. While the calculations here given apply only to the two forms as represented in Fig. 1, the results will be but little different for walls similar in form but which vary considerably therefrom.

Form (a).—The height is h and the bottom width l. The batter of the front face will be taken at 1:12, and the top width at 1/6 of the bottom width. The weight of the masonry will be assumed at 150, and that of the earth filling at 100 lbs/ft³. It will be assumed that the fluid pressure acts against a vertical plane FC; the stability of the entire volume to the left of this plane, including the weight of the earth, will be determined.



FORM (a) FORM (b)

FIG. 1.

Let W_1 denote the weight of masonry per lineal foot, and W_2 the weight of the earth filling to the left of FC. Let P denote the resultant fluid pressure acting at a distance $\frac{1}{3}h$ above the base. Let p denote the weight per cubic foot of such fluid. Then $P=\frac{1}{2}ph^2$.

Assume that the resultant pressure due to the weight of the wall W_1, the weight of the earth W_2, and the pressure P, intersects the base at the edge of the middle third. Equating moments about this point we derive the relation

$$p=132\frac{l^2}{h^2}. \qquad (1)$$

Form (b).—In this form the only forces to be considered are the weight W_1 and the pressure P. Equating these as before there results

$$p=150\frac{l^2}{h^2}. \qquad (2)$$

If the top width in form (a) be made zero the effect on the result would be to change the coefficient in Eq. (1) from 132 to 127, thus showing that a considerable variation in top width has little effect on the result.

Substituting various values of l/h in Eqs. (1) and (2) we have the following values of p, or equivalent fluid weight, under which the wall is stable as above assumed.

	l/h	p
Form (a)	1/3	14.7 lbs/ft³
	4/10	21.1 "
	1/2	33.0 "
Form (b)	1/4	9.4 "
	1/3	16.7 "
	4/10	24.0 "

According to these calculations a fluid weight of 20 to 25 lbs/ft³ may be taken as a basis of design to secure stability equivalent to the ordinary wall, assuming the resultant pressures to cut the edge of the middle third and counting the weight of earth vertically above the back slope as part of the resisting load. It is to be noted that the pressures herein determined are not necessarily the actual earth pressures; the results are to be used only as a means of securing stability of reinforced walls approximately equal to that of solid walls of known proportions.

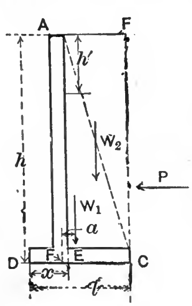


FIG. 2.

262. Stability of Reinforced Concrete Walls.—Fig. 2 represents in outline the usual type of reinforced wall. It consists of a vertical wall AE attached to a floor DC. For low walls the upright part AE may act simply as a cantilever; and likewise the parts EC and ED. For larger walls the part AE is tied to EC at intervals by back walls ACE in the form of narrow transverse walls with tension reinforcement. The projecting portion ED may still act as a cantilever, or it, also, may be connected to the vertical

wall AE by means of buttresses. In either case the earth pressures act in essentially the same manner and the necessary width of base is found in the same way.

Let l = width of base;
x = distance from toe to back of wall AE;
h = height;
p = equivalent fluid weight as determined in Art. 261;
w_2 = weight of earth filling per cubic foot;
W_1 = weight of masonry per lineal foot;
W_2 = weight per lineal foot of earth above the floor EC;
a = lever-arm of W_1 about point F, the edge of the middle third;
P = total fluid pressure $= \frac{1}{2}ph^2$.

Then equating moments about the point F we have

$$W_1a+W_2\left(\tfrac{2}{3}l-\frac{l-x}{2}\right)=\frac{Ph}{3}, \quad \ldots \ldots \quad (3)$$

or

$$W_1a+w_2h(l-x)\left(\tfrac{2}{3}l-\frac{l-x}{2}\right)=\frac{ph^3}{6}. \quad \ldots \ldots \quad (4)$$

If the wall AE is placed well towards the front the moment of the masonry will be small. Neglecting this term and putting $x=kl$ we may solve for l, getting

$$l=h\sqrt{\frac{p}{w_2(1+2k-3k^2)}}. \quad \ldots \ldots \quad (5)$$

This is a minimum for $k=\frac{1}{3}$, that is, for $x=\frac{1}{3}l$. With this value of k we have

$$l=.87\sqrt{\frac{p}{w}}.h. \quad \ldots \ldots \quad (6)$$

For $w=100$

$$l=.087\sqrt{p}.h. \quad \ldots \ldots \quad (7)$$

If, for example, the value of p be taken at 21.1, corresponding to a value of $l/h=4/10$ for a solid wall, the value of l is

equal to $.087\times\sqrt{21.1}\times h=.4h$, or the same as the width of the solid wall.

As it may be desirable to use a smaller or larger value of x than $\frac{1}{3}l$, Table No. 32 has been prepared giving the values of l/h for various values of x/l and various values of p. An examination of the table shows plainly that the length of the projection x makes very little difference in the required total length of base. However, with x made very small or very large the weight of the wall should be taken into account. A further fact brought out by the table and by the table of Art. 261 is that the stability of the reinforced wall is about the same as a solid wall of form (a) shown in Fig. 1 and having the same base length.

TABLE No. 32.

PROPORTIONS OF REINFORCED-CONCRETE RETAINING-WALLS. (See Fig. 2.)

VALUES OF l/h FOR DIFFERENT VALUES OF p AND FOR $w_2=100$ (EQ. (5)).

Values of $k=x/l$	Values of Equivalent Fluid Weight p. Pounds per Cubic Foot.			
	15	20	25	33
.5	.35	.40	.45	.51
.33	.34	.39	.43	.50
.25	.34	.39	.44	.50
.20	.34	.40	.44	.51
.15	.35	.40	.45	.52
.10	.36	.41	.46	.53
0	.39	.45	.50	.57

The resultant forces acting upon the three parts of the wall AE, DE, and EC must be determined. On the wall AE the force may be taken as a horizontal force equal to $P_1=\frac{1}{2}ph^2$, and applied a distance $\frac{1}{3}h$ above the base. The resultant force acting on any length h' from the top is likewise $\frac{1}{2}ph'^2$ and applied a distance $\frac{2}{3}h'$ below the top. The pressure on the foundation will equal the total weight W_1+W_2 and will be applied a distance $\frac{1}{3}l$ from point D. The average unit pressure will be

$(W_1+W_2)/l$, and the maximum pressure at D will be twice this value.

The upward pressure under the cantilever DE will vary from a maximum of $2\frac{W_1+W_2}{l}$ at D to a value under the point E of $2\frac{W_1+W_2}{l}\times\frac{l-x}{l}$. This is a "trapezoid" of pressure, and where x is large the centre of gravity of the trapezoid may be found and the resultant applied at this point. Usually it will be accurate enough to assume the pressure on DE as uniformly distributed at an average value and applied at the centre of the projection outside of the vertical wall.

The upward pressure on the floor EC varies from the value above given at E, to zero at C. It varies uniformly between these points. The downward pressure is the weight of the earth above the floor, $=W_2$. This may be assumed as uniformly distributed and equal to w_2h per unit area at all points. The total downward pressure on EC will be greater than the upward pressure unless x is very small.

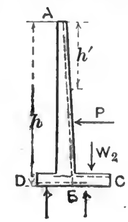


FIG. 3.

263. Design of Wall.—In discussing the design it will be necessary to consider two forms: (1) the cantilever wall without back tie-walls as in Fig. 3, and the wall provided with such back walls as in Fig. 4.

The form of Fig. 3 is adapted to heights of about 12 to 18 feet. For high walls the form of Fig. 4 will be more economical.

Form (*a*). (Fig. 3.)—The maximum moment in the upright portion AE is $P\frac{h}{3}=\frac{ph^3}{6}$. At any distance h' below the top the moment is $\frac{ph'^3}{6}$. Only a portion of the reinforcing-rods need be carried up the full height. The shear at the bottom is $P=\frac{ph^2}{2}$. This will be very small and will require no

special attention. The reinforcing-rods of a cantilever beam have their maximum stress at the end of the beam, hence special care must be given to secure an effective bond or anchorage. In the figure the vertical rods have an insufficient length below the point of maximum moment to develop their full strength, and therefore they should be anchored in a substantial manner. This may be done by screw-ends and nuts, or by looping the rods around anchor-bars near the bottom of the floor *DC*.

The cantilever *DE* must be treated in the same manner as the upright cantilever. The pressures will be much heavier and the shear and bond stress may need attention. The reinforcement should extend far enough beyond *E* for bond strength.

The cantilever *EC* is acted upon by an upward and a downward force as shown in the figure. The maximum moment will be at *E* and will be negative. It is provided for by reinforcement as shown.

To secure maximum economy each one of the cantilevers may be tapered towards the end to a minimum practicable thickness. The bending moments at various sections in a cantilever beam uniformly loaded vary as the squares of the distances from the free end. The resisting moments vary approximately as the squares of the depths of the beam. Hence a beam tapering uniformly to zero depth at the end would be of the necessary depth at all points. The moments in the vertical beam *AE* vary as the cubes of the distances below the top, so that a straight taper will in this case give a beam whose weakest point will be at the bottom. At the top point *A* some form of coping is usually added, of a width according to the requirements of the case.

To prevent unsightly cracks a certain amount of longitudinal reinforcement is necessary. The amount required per square foot of cross-section will be less the heavier the wall, as temperature changes will be less in such a wall. On the basis of the discussion in Chap. II, Art. 42, the percentage required may be placed at about 0.4% as a maximum for thin walls, to

perhaps one-half of this for heavy walls. High elastic-limit material is advantageous for this purpose.

Form (*b*). (Fig. 4.)—So far as the external pressures are concerned they have been explained in Art. 261 and are practically the same as in the previous case considered. The loads or pressures on the concrete are, however, carried quite differently. The toe *DE* is the same as in form (*a*) and reinforced in the same way. The pressure against the longitudinal wall *AE* is carried laterally for the most part and given over to the inclined

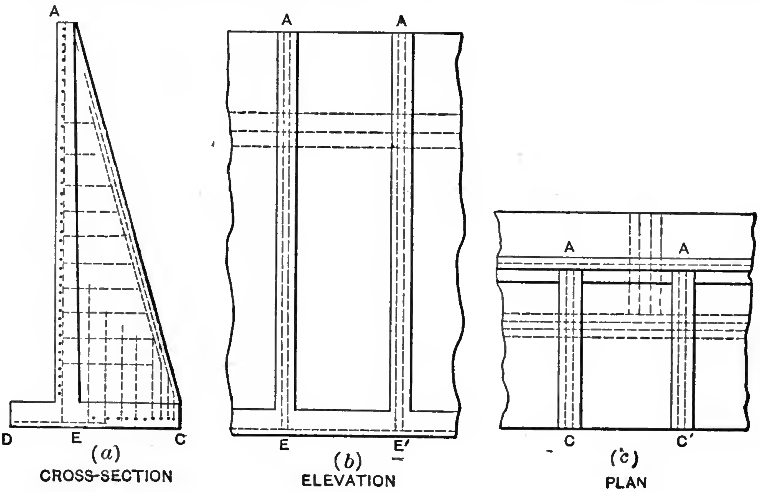


FIG. 4.

back walls. The wall *AE* must therefore be designed as a slab supported along the lines *AE* and *A′E′* (Fig. 4 (*b*)), and subjected to a pressure per square foot at any point a distance h' below the top equal to ph'. Near the bottom, the load on *AE* is transmitted more or less to the floor *EC*. The wall should therefore be bonded to the floor with a small amount of vertical reinforcement, which may well extend to the top to prevent cracks, although under ordinary conditions the wall *AE* is under some vertical pressure.

The floor *EC* is subjected to both upward and downward pressures, the latter exceeding the former towards the end *C*,

and possibly throughout, as previously explained. This floor is supported by the back wall AEC and is therefore reinforced longitudinally as a floor-slab in accordance with the resultant pressure at any point. Here, again, it is well to bond the floor to the wall AE by extending the transverse reinforcement of the toe DE into the portion EC.

The back wall ACE acts as a cantilever beam anchored to the floor. It is also a T-beam, the flange being the longitudinal wall AE. The tension along the edge AC is carried by rods near this edge, whose stress at any point is found with sufficient accuracy by an equation of moments taken about the center of the front wall. The maximum stress will be at the bottom, if the wall is made with a straight profile. At the connection of the wall AEC to the floor, it is to be noted that the floor load is transferred to the wall along the line EC, but mainly near the end C. The main tension-rods in AC should therefore be distributed somewhat at their lower ends and well anchored to the reinforcing-rods of the floor EC. A few additional vertical rods should also be put in to insure thorough bonding of floor to wall. These will also carry a part of the tension in the back wall, but will not be as efficient as the rods nearer the outside edge. It is desirable, likewise, to bond the vertical wall AE to the back wall with short horizontal rods as shown. The slabs formed by the walls AE and the floor EC are continuous over supports, and if the span is long should be provided with some reinforcement for negative moments at these supports.

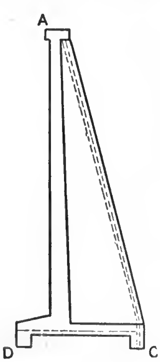


FIG. 5.

Fig. 5 shows some additional features of design which have been used. A longitudinal beam is built at C and the floor is thus supported on all four edges. The main rods along AC are then anchored into the beam.

A horizontal beam may also be made of the coping at A, thus giving some support to the wall AB along its upper edge.

A projection may be necessary at the toe *D*, or elsewhere, in order to increase the resistance against forward sliding. The beam *C* aids in this respect.

264. Illustrative Examples. — Fig. 6 shows the form of retaining-wall used on the Great Northern R. R. at Seattle,

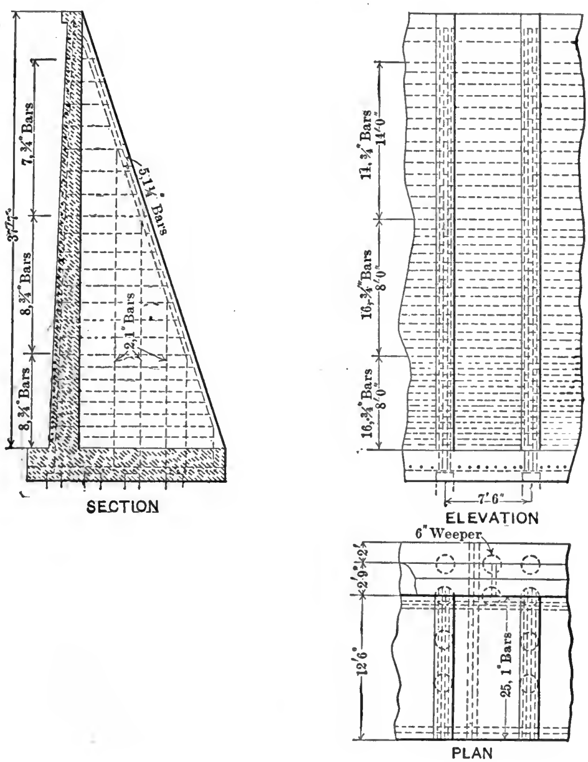


Fig. 6.—Retaining-wall, Great Northern Railway.

Wash.* This is a good illustration of the second type above discussed. An estimate by Mr. C. F. Graff of the amounts of material per lineal foot required in reinforced and plain con-

* Eng. News, Vol. LIII, 1905, p. 262.

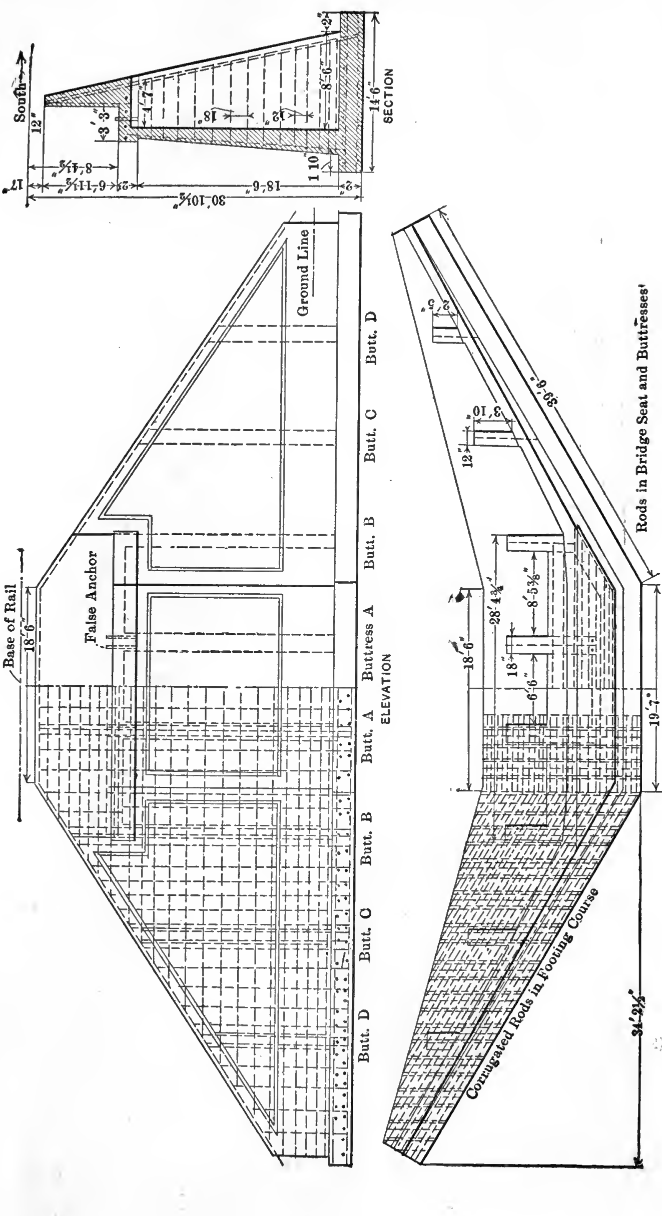


FIG. 7.—Standard Abutment, Wabash Railroad.

crete walls, made in connection with the design of Fig. 6, gave the following results:

Height of Wall, Feet.	Amount of Concrete per Lineal Foot.		Saving: Per Cent of Reinforced Wall.
	Plain Wall, Cubic Feet.	Reinforced Wall, Cubic Feet.	
40	396.4	218	45
30	226	127.8	43.3
20	110	69.9	36.4
10	44	34.9	20.4

The steel was included by adding its concrete equivalent.

Fig. 7 illustrates a standard form of abutment used by the Wabash R.R. Co.*

265. Retaining-walls Supported at the Top.—Frequently a retaining-wall may be supported at the top. In such a case it is designed as a simple beam supported at the top and bottom; or vertical ribs or beams may thus be calculated and the slab reinforced horizontally and supported by these ribs.

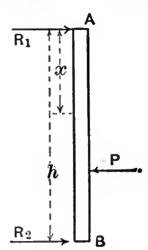


Fig. 8.

A wall AB (Fig. 8) acted upon by a pressure uniformly varying from zero at the top to a maximum at the bottom will be subjected to a bending moment whose maximum value will be determined. Let the pressure be that due to a fluid weighing p lbs/ft³. Then $P=\frac{1}{2}ph^2$, $R_1=\frac{1}{3}P=\frac{1}{6}ph^2$, $R_2=\frac{1}{3}ph^2$.

The bending moment M at a distance x below A is

$$M=R_1x-\tfrac{1}{6}px^3=p/6\,(h^2x-x^3).$$

This is a maximum for $x=h\sqrt{\frac{1}{3}}=.58h$. The maximum moment is then

$$M=.064ph^3. \qquad (8)$$

If the pressure is water pressure, as in a reservoir, the value of the maximum moment becomes equal to

$$M=4h^3, \qquad (9)$$

where the units are the foot and pound. For an earth retaining-wall with $p=20$, then $M=1.3h^3$, etc.

* Ry. Rev., Vol. XLV, 1905, p. 523.

DAMS.

266. The dam is a form of retaining-wall, but is subject to somewhat different conditions as to pressures. For this case a form of wall as shown in Fig. 9 is poorly adapted, owing to the fact that the water pressure will probably penetrate beneath the floor *DC* and exert an upward force nearly equal to the downward pressure, thus destroying the usefulness of the floor *EC*. To obviate these objections the wall *AE* must be brought back to the point *C*. Increased stability will then be secured by making it inclined. In this position, it will naturally be supported by transverse walls or buttresses, resting on a floor *DC*, or directly on the foundation material, as shown in Fig. 10. The water pressure on the floor may then be relieved by

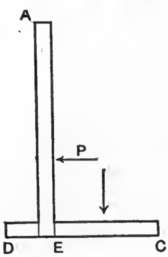


FIG. 9.

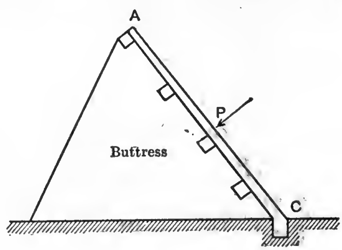


FIG. 10.

drain-openings allowing free exit for seepage-water. Thus built it forms a stable and efficient type of dam. Its design as to stresses and sections is simple and obvious. The wall or floor *AC* may be supported directly on the cross-walls and reinforced with longitudinal rods, or longitudinal beams may be used as shown and the slab supported on these. The pressure on the foundation is determined by considering the resultant of water pressure and weight of dam. The buttresses or cross-walls are subjected only to compressive stresses. Ample longitudinal reinforcement should be provided to thoroughly bind the structure together. Dams are often subjected to dynamic loads as well as static pressures, and sections must be provided more liberally than in many other structures.

The form shown in Fig. 10 is not suited to act as a spillway except for low falls. For a spillway the down-stream edge of the buttresses is also covered with a floor which may be curved in the usual manner.

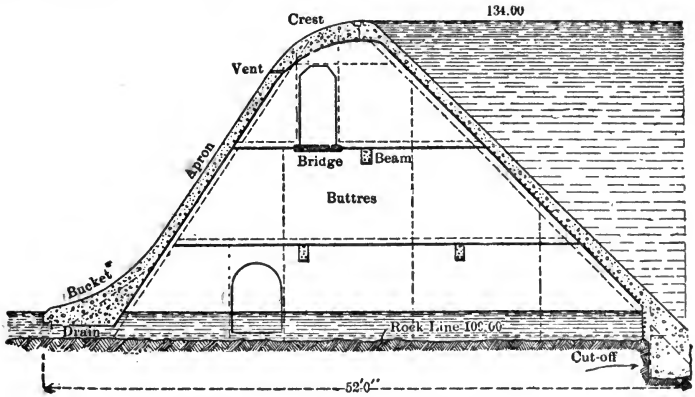


Fig. 11.—Dam at Schuylerville, N. Y.

Fig. 11 illustrates a dam of this type built at Schuylerville, N. Y., by the Ambursen Hydraulic Construction Co.* A foot-way is provided for in the interior. The design as to strength is obvious.

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* Eng. News, Vol. LIII, p. 448.

CHAPTER XII.

MISCELLANEOUS STRUCTURES.

GIRDER BRIDGES AND CULVERTS.

267. For short spans, the girder bridge or box culvert is likely to be a more economical form than the arch, owing to the less rigid requirements for foundations and abutments. For purposes of analysis this type of structure may be divided roughly into three classes: (1) Simple spans in which the girder rests upon independent abutments or piers; (2) concrete trestles or bridges in which the girders, abutments and piers form a monolithic structure; and (3) pipe culverts and box culverts built as square or rectangular pipes.

268. **The Simple Beam Bridge.** — These are designed in the same manner as any other concrete floor. Spans up to 20 to 30 feet may well be made as a simple slab or uniform thickness spanning the opening. For railroad structures the loads are relatively so large that shearing stresses will usually require careful attention. For longer spans a gain in economy will result by the use of main horizontal girders of relatively great depth, with a floor supported by the girders and reinforced transversely. The bridge may be made either a "through" or "deck" girder, according to the requirements of the case, the latter being the more economical. Floors of reinforced concrete are also used for steel truss and girder bridges to a considerable extent where a solid floor is desired. The details are arranged in a variety of ways, but the calculation and design of the reinforcement to meet the given conditions require no special consideration. The proper allowance for impact is an

important point in this connection. Durability is an important factor favorable to the use of reinforced concrete for bridge floors.

269. Concrete Trestles. — Where several short spans are required and concrete is used for both the girders and the piers, the latter may usually be made of comparatively small cross-section, — much smaller than possible if ordinary masonry be used. The structure then approaches the ordinary floor and column construction in the relations of its parts. The piers, if lightly loaded, may consist merely of two or more columns connected by a suitable portal. In some extreme cases designs have been carried out in which the supporting piers or towers have been arranged in a manner similar to a steel trestle, even to the diagonal bracing. It would seem, however, that the treatment of concrete should be on somewhat different lines than is best suited to such a material as steel, and that structural forms in concrete should be somewhat massive and limited in general to the beam and the compression member.

Where the piers are made small, as here assumed, they must be built rigidly in connection with the girders of one or more spans, as are the columns in a building. The girders must be designed with proper reference to their continuity, and the piers must be able to resist a certain amount of bending moment. This moment can be estimated in the manner suggested in Chapter IX, Art. 237.

As an example, let Fig. 1 represent a concrete trestle of monolithic construction. The girders are continuous and the

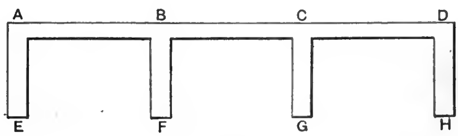

Fig. 1

piers are rigidly attached to them. The greatest moment in the pier BF will occur when one of the spans AB or BC is loaded.

Suppose BC be loaded. Then calculate the negative moment at B, assuming BC to be fixed at the ends. This moment will be equal to $-\frac{1}{12}pl^2$, where p = load per foot and l = span length. Now this moment is distributed at the joint B among the three members AB, BF, and BC in proportion to the value of I/l for the three members, the length l being taken as the estimated length to the point of inflection in each case (the full length of BF). This will determine approximately the moment in BF. The *maximum* negative moment in BC and AB will occur when both spans are loaded and will be approximately equal to $\frac{1}{10}pl^2$. (See Chapter IX, Art. 218.) The end piers or abutments must be designed also as retaining walls.

270. Pipe and Box Culverts. — For small openings the monolithic pipe or box form is very advantageous. This form of structure is a complete opening in itself and so long as intact will do good service. Considerable settlement, as a whole, may be permissible, and hence solid foundations may not be needed.

The cross-section may be circular, elliptical or rectangular. Theoretically, the elliptical form is the best as corresponding more nearly to the requirements for resisting the earth pressure. The circular is practically as good for small openings, while for large openings the rectangular form will often be the best on account of its simplicity and the lesser head room required. Where the culvert is manufactured at a shop and transported to the site, the circular or elliptical forms will usually be the most advantageous. As the loads coming upon such structures are not accurately known an exact analysis of the stresses is impossible, but the results obtained for certain simple cases will be useful as a guide to the judgment. The general method of analysis employed in Chapter X has been used. The details of the analysis will be omitted.

271. *The Circular Culvert.* — Two cases have been analyzed; (1) for a uniform load, and (2) for a concentrated load.

Case I, Uniform load. (Fig. 2.) It is assumed that the pressure on the pipe is exerted in parallel lines (as downward and upward) and is uniformly distributed with respect to a plane perpendicular to the direction of the pressure.

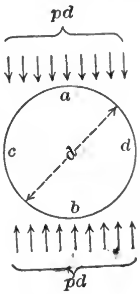

FIG. 2.

Let d = diameter of pipe;
p = pressure per unit area as measured perpendicularly to the pressure;
M = bending moment in pipe in a length of one unit;

Then the following equations result.

$$M_a = M_b = \tfrac{1}{16}\,pd^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$M_c = M_d = -\tfrac{1}{16}\,pd^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the lateral pressure, measured in a similar way, be p' per unit area, then the moments due to this pressure will be

$$M_a = M_b = -\tfrac{1}{16}\,p'd^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$M_c = M_d = \tfrac{1}{16}\,p'd^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For equal horizontal and vertical forces (equivalent to a uniform radial pressure), the moments at all points are zero. Usually the lateral pressure will be much less than the vertical pressure; probably not more than one-fourth or one-fifth as much. Assuming a ratio of one-fourth, the resulting total bending moments at the points a, b, c, d, will be $\tfrac{3}{64}\,pd^2$, positive at the top and bottom and negative at the sides.

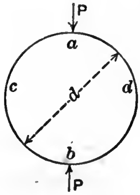

FIG. 3.

Case II; Concentrated loads at opposite points (Fig. 3.)

In this case the moments are

$$M_a = M_b = .16\,Pd \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$
$$M_c = M_d = -.09\,Pd \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

272. *The Rectangular Culvert.*—Case I; Uniform loads (Fig. 4).

Let l_1 = width of culvert;
l_2 = height of culvert;
I_1 = moment of inertia of top and bottom, assumed as equal;
I_2 = moment of inertia of sides;
p = vertical load and foundation reaction per unit area.

Then

$$M_a = M_b = \frac{pl_1^2}{8} \cdot \frac{\frac{1}{3} l_1/I_1 + l_2/I_2}{l_1/I_1 + l_2/I_2} \quad . \quad . \quad . \quad . \quad (7)$$

$$M_c = M_d = M_a - \tfrac{1}{8} pl_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The moments at e and f are equal to M_c.

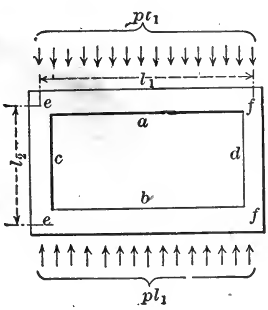

FIG. 4.

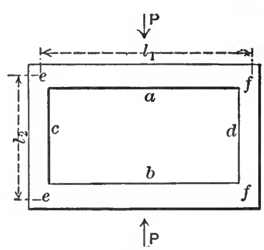

FIG. 5.

For a square culvert with uniform section $M_a = \frac{1}{12} pl^2$ and $M_c = -\frac{1}{24} pl^2$.

For equal vertical and lateral loads the moments in the square culvert become $M_a = M_c = +\frac{1}{24} pl^2$ and $M_e = -\frac{1}{12} pl^2$ as in a beam with fixed ends.

Case II; Concentrated loads. (Fig. 5.)

For vertical loads applied centrally,

$$M_a = M_b = \frac{Pl_1}{4} \cdot \frac{\frac{1}{2} l_1/I_1 + l_2/I_2}{l_1/I_1 + l_2/I_2} \quad . \quad . \quad . \quad . \quad (9)$$

$$M_c = M_d = M_a - \tfrac{1}{4} Pl_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For the square form, $M_a = \frac{3}{16} Pl_1$ and $M_c = -\frac{1}{16} Pl_1$; and for equal lateral and vertical forces $M_a = M_c = \frac{1}{8} Pl_1$ and $M_e = -\frac{1}{8} Pl_1$ as for fixed beams.

273. Arrangement of Reinforcement.—The bending moments here determined are based on the assumption that the entire section is reinforced so as to act as a monolithic structure. This of course requires proper reinforcement for negative as well as positive moments.

In the circular form a wire mesh is convenient, especially for small diameters. A single mesh will be sufficient, placed near the intrados at top and bottom and near the extrados at the sides, crossing the central axis at about the quarter point.

In the rectangular form, if reinforcement for negative moments at the corners is omitted, then the four sides will act as simple beams, the concrete cracking more or less on the outside near the corners.

Longitudinal reinforcement should be provided to some extent. Where foundations are good a very small amount will be sufficient, but if settlement is likely to occur the longitudinal reinforcement becomes of much importance. The entire culvert will act as a beam subjected in the main to positive bending moments. Most of the reinforcement should therefore be placed along the bottom of the culvert.

274. Tests of Reinforced Concrete Rings and Culvert Pipe.—Large reinforced concrete rings and pipe have been tested by Professor Talbot, with results agreeing closely with the theoretical analysis of Art. 271. Loads were applied in two ways, (*a*) as concentrated loads, in which the pipe was supported along an element at the bottom and the load was applied along an element at the top; and (*b*) as distributed loads, in which the pressures at bottom and top were distributed as uniformly as possible over the entire horizontal projection of the pipe by means of a carefully constructed sand box.* These

* For full details see Bulletin No. 22, Eng. Exp. Sta., University of Illinois, 1908.

methods of loading correspond to the cases of concentrated and distributed loads discussed in Art. 271. All rings and pipe were 48-in. internal diameter and 4-in. thick, the rings being 24 in. long, and the pipe sections 102–104 in. long. The latter were made with the usual bell end. The reinforcement consisted of $\frac{1}{4}$-in. corrugated bars except in No. 982, in which $\frac{1}{2}$-in. bars were used, and in No. 988, in which No. 3 Clinton wire mesh was used. The rings were made circular and the reinforcement placed near the intrados at top and bottom and near the extrados at the sides. In the pipe, the reinforcement was made circular and the concrete cast with a vertical diameter 4 in. greater than the horizontal, giving a similar relative position for the reinforcement. The results are given in Tables 33 and 34. In these tables the value of t is the net or effective thickness of the pipe as measured to the center of the steel. In the last column of Table 33 the theoretical strength is the strength calculated by means of eq. (5), Art. 271, assuming the bending resistance of the pipe to be .87 Af_st per lineal foot, using the yield-point of the steel for the value

TABLE No. 33.

RESULTS OF TESTS ON REINFORCED CONCRETE RINGS.

(TALBOT.)

(Concentrated Loads.)

Diam. of rings = 48 ins.; thickness = 4 ins.; age 1–3 mos.

No.	Reinforcement, Per Cent.	Load at First Crack, Lbs/lin.ft.	Maximum Load, Lbs/lin.ft.	t inches.	Ratio of Theoretical to Actual Strength.
926	0.73	1500	2850	2.75	1.12
928	0.80	1400	3550	2.5	0.82
931	0.73	2150	2500	2.75	1.28
932	0.66	1500	3000	3.0	1.16
933	1.00	1170	3170	2.0	0.73
934	0.80	1300	3150	2.5	0.92
952	1.00	1000	2350	2.0	0.99
953	0.89	1200	3600	2.25	0.73
971	0.73	1500	4120	2.75	0.77

TABLE No. 34.

RESULT OF TESTS ON REINFORCED CONCRETE RINGS AND PIPES.

(TALBOT.)

(Distributed Loads.)

Diam. of rings and pipe = 48 ins.; thickness = 4 ins.; age of rings 1–3 mos.; age of pipe 4–6 months.

REINFORCED CONCRETE RINGS.

No.	Reinforcement, Per Cent.	Load at First Crack, Lbs/lin.ft.	Critical Load, Lbs/lin.ft.	Maximum Load, Lbs/lin.ft.	t inches.	Ratio of Theoretical to Actual Strength at Critical Load.
923*	0.80	2250	7000	10500	2.5	1.06
921	0.80	3500	10000	23500	2.5	0.74
922	0.80	3250	10000	18500	2.5	0.74
927	0.80	3250	8000	26000	2.5	0.93
951	0.80	3200	9000	25000	2.5	0.83
972	0.73	4500	8000	17500	2.75	1.03
976	0.66	4000	9000	19000	3.0	0.99
977	0.66	4000	10000	21000	3.0	0.89

REINFORCED CONCRETE PIPE.

No.	Reinforcement, Per Cent.	Load at First Crack, Lbs/lin.ft.	Critical Load, Lbs/lin.ft.	Maximum Load, Lbs/lin.ft.	t inches.	Ratio of Theoretical to Actual Strength at Critical Load.
981	0.66	8360	19500	31500	3.0	0.55
982	1.39	10960	15000	24800	3.0	1.49
983	0.66	4950	12500	23800	3.0	0.86
988	0.88	6700	9000	31400	3.0	

* No lateral restraint.

of f_s. This was taken at 46,400 lbs/in² for the rings and 55,000 lbs/in² for the pipe. The tests showed that the primary cause of failure was the failure of the steel in tension. In Table 34, similarly, the theoretical strength is determined on the basis of eq. (1) of Art. 271. In the case of the distributed load tests the "critical load" was estimated as the load beyond which the increased resistance was primarily due to increased lateral resistance of the sand filling and not of the ring or pipe itself. The ultimate resistance would obviously be chiefly dependent upon the character of the filling about the pipe.

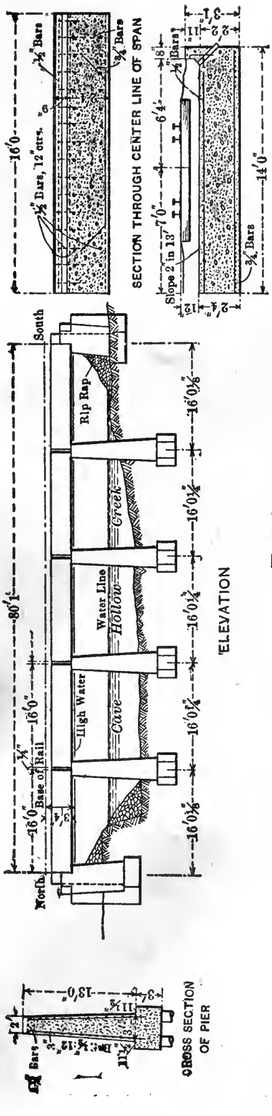
SECTION THROUGH CENTER LINE OF SPAN
ELEVATION
CROSS SECTION OF PIER
Rip Rap
Water Line
High Water
Cave Hollow Creek
Base of Rail
North
South
80′1″
16′0″
14′0″
Slope 2″ in 13″

FIG. 6.

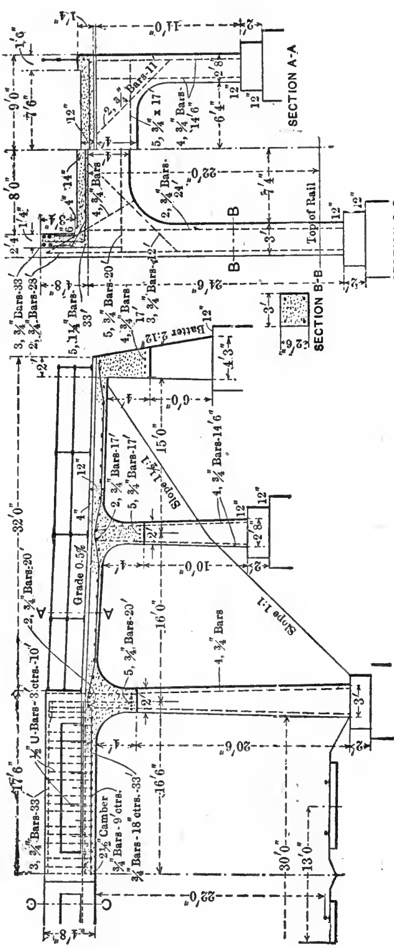
SECTION A-A
SECTION B-B
SECTION C-C
Top of Rail
Grade 0.5%
Slope 1:1
Batter 2:12
21/2″ Camber
32′0″
22′0″
20′6″
30′0″
13′0″
17′6″

FIG. 7.

The results of the concentrated load tests show close agreement between the theoretical and experimental values. This means merely that the resisting moment of a pipe may be taken the same as that of a straight beam of the same thickness and reinforcement. The tests under distributed loads show that the theoretical strength can be secured by very careful bedding and testing. It would seem that in practice nothing should be allowed for lateral support and that the theoretical moment of $1/16\, pd^2$ may be used if the filling and bedding are carefully done. If poorly done a larger bending moment will exist and will need to be considered.

275. Illustrative Examples.—Fig. 6 illustrates a simple beam bridge or "trestle" on the Chicago, Burlington and Quincy R.R.* The girder consists of a slab twenty-four inches in thickness, reinforced as shown in the illustration. The piers are separate structures.

Fig. 7 represents a concrete highway bridge as an overhead crossing of the Big Four R.R. This design illustrates the deep girder with floor-slab reinforced transversely, and also the "trestle" in which the piers are columns built as one piece with the girders.†

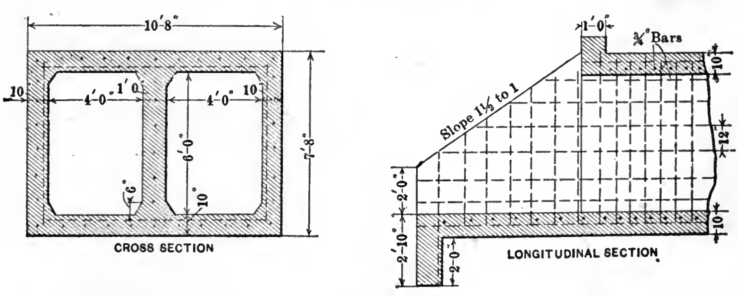

FIG. 8.

Fig. 8 illustrates a standard design for a monolithic box culvert. It is not reinforced for negative moment at the

* R. R. Gaz., Vol. XL, 1906, p. 713.
† R. R. Gaz., Vol. XL, 1906, p. 497.

corners. This form of construction is applicable to many other structures as subways, tunnel linings, etc. No special consideration of these various applications of the reinforced beam is required in this place. A clear understanding of the general principles of reinforced concrete design will enable the details to be suitably modified to meet the conditions of the case.

CONDUITS AND PIPE LINES.

276. For conduits not under pressure, large sewers and the like, reinforced concrete lends itself to convenient and economical construction.

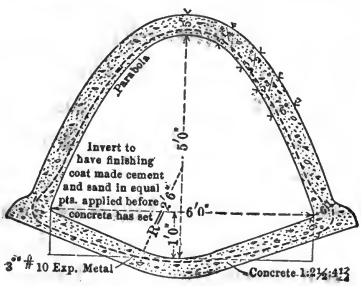

FIG. 9.

As to the analysis and design, these structures are only special cases of the monolithic pipe or box discussed in preceding articles. The character of the foundation

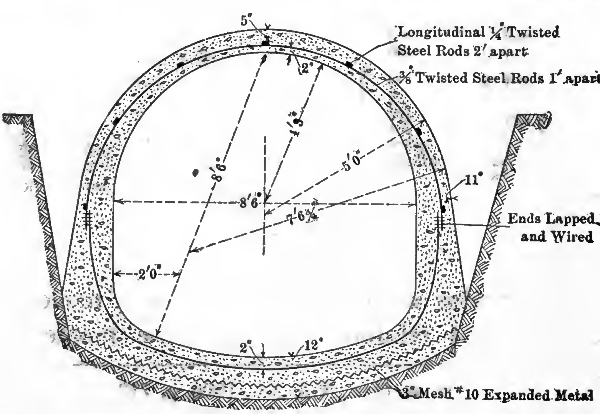

FIG. 10.

and convenience in construction will lead to various modifications of design.

Fig. 9 is a typical cross-section of a large sewer for Harrisburg, Pa. A mesh of expanded metal is used for reinforcement, arranged to resist positive moments excepting at bottom and corners.

Fig. 10 illustrates a large conduit of the Jersey City Water Supply. This section is employed where the bottom is soft, special reinforcement being used in the invert. The position of the reinforcement to carry positive moments at crown and negative moments at sides should be noted.

Reinforced concrete has also been used to some extent for pipes under pressure, but it is very difficult to secure imperviousness under heads of considerable magnitude. In pressure pipes the tensile stress is entirely taken by the steel, the concrete furnishing merely the impervious layer and resisting bending due to earth loading.

Tanks, Reservoirs, Bins, etc.

277. For covered reservoirs reinforced concrete is very well adapted. The rectangular form with flat cover is usually the most convenient; its design involves the same features as building design with the additional one of imperviousness. Elevated towers and tanks may also be made of concrete, but high pressures are difficult to deal with.

Bins and coal pockets are structures for which concrete is well adapted. For the storage of coal unprotected steel is not durable, but reinforced concrete furnishes an almost ideal material, lending itself readily to the necessary form for strength and furnishing the desired durability.

Reinforced concrete is advantageously used in other minor forms of structures and structural elements. Noteworthy among such uses are its employment for piles, fence posts, and poles for various purposes. For piles it is especially advantageous in situations where continual submergence is not certain.

CHAPTER XIII.

REINFORCED-CONCRETE CHIMNEYS.

278. General Description.— Design and construction of reinforced-concrete chimneys has kept pace with that of other structures made of that material, and they are now being built to meet any and all conditions of service. For several recent years (1916–1918), the tallest chimney in the world (570 ft. high) was one of reinforced concrete. A reinforced-concrete chimney, like a brick one, consists of a base, a shaft or outer shell, and generally a lining or inner shell.

The principal reinforcement of the outer shell consists of vertical rods in the concrete, the purpose of which is to withstand tensile stress induced in the windward side by strong winds. This ability to withstand large tensile stress due to wind is a principal advantage possessed by the reinforced concrete chimney over a brick one. To meet this shortcoming, a brick chimney must be built so heavy that strong winds produce little or no tension in the shells. As a result outer concrete chimney shells are much lighter than brick ones. For example, the tall chimney above mentioned has a shell 7 in. thick at the top and $29\frac{1}{2}$ in. at the base, whereas the brick chimney just overtopping it, and slightly less in average diameter, has a shell $13\frac{1}{2}$ in. thick at top and 5 ft. 1 in. at base. Circumferential steel is provided to reinforce against temperature cracks (vertical) and tension and shear.

Bases are generally square or octagonal in plan, and thinner at the perimeter than at the shells; but small bases have been built uniform in thickness for simplicity, and large ones have been built thin near the center as well as at the perimeter to effect a saving of concrete. They are generally reinforced with a criss-cross of steel rods.

Linings are provided for protection of the outer shell from hot flue gases. For ordinary boiler purposes linings are gener-

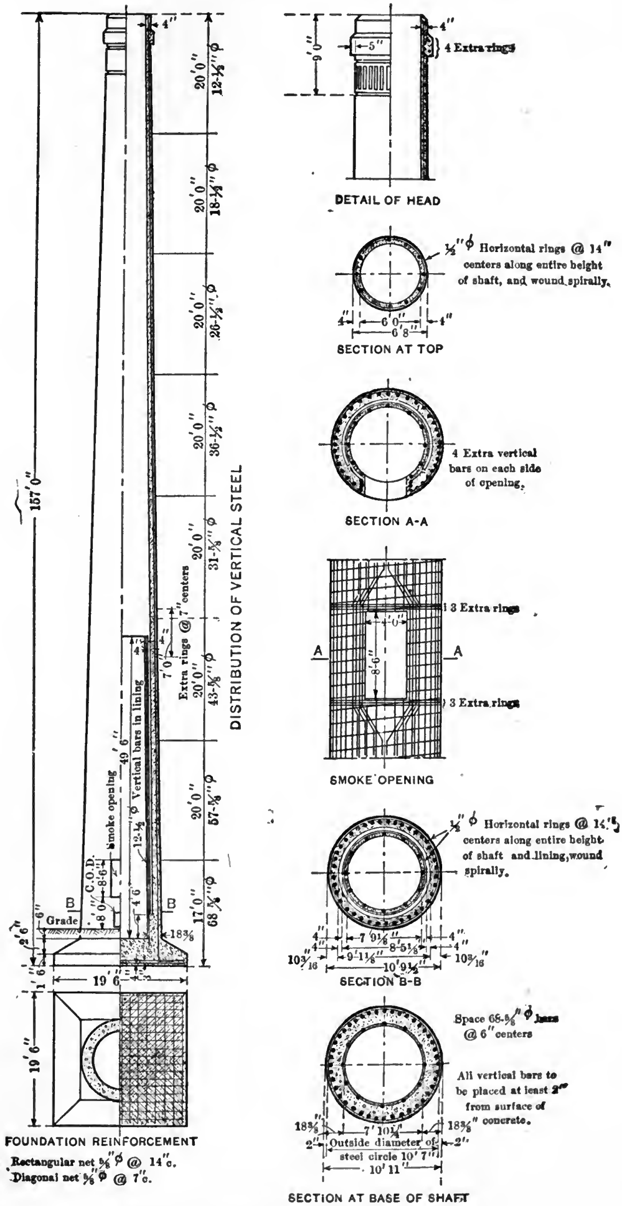

FIG. 1.—Design of a 157-ft. Chimney.

ally built to extend about one-third the height of the chimney; for open hearth furnaces, kilns, etc., they are built higher, even to the top. The thickness is generally 4 or 5 in. and the air space between the shells is made as small as the forming will allow, about 4 in. The air space communicates with the outer air through vent holes. There is no rigid connection between the shells except at the base, hence the lining is free to expand and contract and is not subjected to wind stress. But they are lightly reinforced vertically and circumferentially to preserve the integrity of the lining even if cracked.

Fig. 1 illustrates a typical design, by the Weber Chimney Construction Company of Chicago.

279. Design of a Chimney.—The inner diameter and the height of the chimney are determined by the requiremeuts of draft and capacity—matters which do not fall within the scope of this work. The outer shell at the top may be made as thin as 4 inches except for the larger chimneys (see Art. 278). Shell thickness at lower levels (or sections) are chosen tentatively, also amounts of reinforcements, both vertical and circumferential, and then various sections are investigated for stress. Methods for computing wind and temperature stresses are explained in Arts. 280–282 and 284–287; for shear or diagonal tensile stress, see Chapter IV.

A base should be made with such an extent of bottom that its greatest pressure on the earth due to weight of chimney, weight of earth filling over the base, and wind pressure, will not exceed the permissible limit; and the base itself must be strong enough to withstand the pressures on its top and bottom. Methods for investigating these points for a given base are explained in Arts. 288–290.

Linings are designed, as yet, by precedent. In the report on his investigations of reinforced-concrete chimneys,* Sanford E. Thompson recommends that for temperatures above 750° F. fire-brick be used for linings, but for lower, suitable cement mortar may be safely used. Inasmuch as a number of outer

* For abstract, see Eng. News, Jan. 9, 1908.

shells have cracked badly near the top of the lining, he suggests that linings be built higher than one-third the height or else that the outer shell be extra-reinforced near the top of the lining. No reports of injured linings were received by him. The usual thickness, 4 in. for moderate heights, and the reinforcements used would seem adequate but it should be noted that comparatively few owners have examined the linings of their chimneys. Published descriptions of reinforcements are meager. In one lining, 90 ft. high, the percentage of hoop steel is $\frac{1}{4}$ and the hoops are spaced 20 in.; in another 60 ft. high, it is $\frac{1}{6}\%$ and the spacing is 36 in. In the latter the percentage of vertical steel is 1.8.

280. **Wind Stresses in the Outer Shell.**—On a horizontal section of a chimney sustaining no wind pressure, the "fiber stress" in the concrete is a uniform compression. Wind pressure changes this uniform stress, increasing the intensity of the compression on the lee side and decreasing it on the windward. The decrease may be larger than the preexistent intensity, the net result being a tensile stress. Two cases will be distinguished; in both it is assumed, just as in the most widely used flexure formulas for the working strength of an ordinary reinforced-concrete beam, that the fiber stress is a uniformly varying one.

Notation.—In this connection see Figs. 2 to 6. Also let

A = area of chimney section under consideration;

A_s = total area of all steel sections there;

W = weight of superincumbent portion of chimney;

P = wind pressure on that portion;

M = bending moment at the section;

e = distance from the center of the section to where the resultant of the weight and wind pressure cuts the section, "eccentric distance";

f_c = unit stress in concrete adjacent to the steel at lee side;

f_c' = unit stress in concrete adjacent to steel at windward side;

f = unit stress in concrete at the lee side;

f' = unit stress in concrete at the windward side;

f_s = unit stress in steel at the windward side;
m = a coefficient such that $f_c = mW/A$;
m' = a coefficient such that $f_c' = m'W/A$;
p = steel ratio, i.e., A_s/A; and
n = ratio of modulus of elasticity of steel to that of concrete (taken as 15 in all numerical work following).

281.—*Case I.* The stress at the windward side is compressive or a tension of low intensity, say, 50 lbs/in.^{2} This case obtains in all sections where the resultant of W and P falls within or not far without the kern* of the section. (The kern is a circle concentric with the hollow circle, its radius being $\frac{1}{4}r_2[1+(r_1/r_2)^2]$. Since in chimneys r_1/r_2 is nearly 1, the kern radius is nearly $\frac{1}{2}r_2$; it may be taken as $\frac{1}{2}r$.) Fig. 3 (*a*) represents the variation in the concrete stress (wholly compressive) when the resultant falls well within the kern, and Fig. 3 (*b*) represents it (part tensile) when the resultant falls outside.

First Method.—This is carried out graphically by means of a diagram (Fig. 2); it will be described by examples, and then the analysis on which the diagram is based will be given. At the base of the diagram there are given values of the eccentricity, e/r from 0 to 0.8; at the left side values of the coefficient m, and at the right values of the coefficient m'.

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* Imagine all the forces acting on either side of a section of a beam, or column, etc., to be compounded into a resultant R, or if that is impossible, into two forces: one, N, perpendicular to the section and one in the section (always possible). If R or N cuts the section at its centroid, then the normal stress at the section is all one kind, tension or compression, and it is a uniform stress. If R or N cuts the section and not very eccentrically, then the normal stress will still be of one kind but it will not be a uniform one. That part of the cross-section within which R or N must cut the section in order that the normal stress may be of one kind, is called the *kern* of the section.

When the normal stress varies uniformly, then the kern for any section can be determined easily. For this kind of normal stress the kern for a hollow circle is described above and for a square an octagon and a circle in Art. 289.

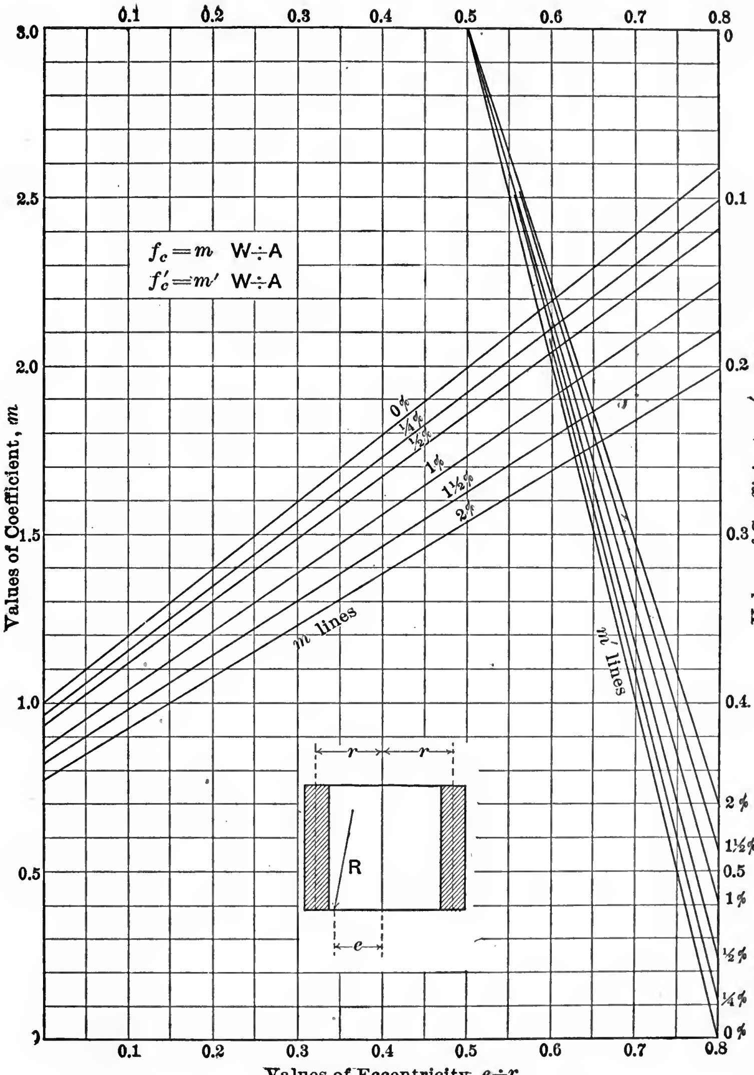


FIG. 2.—Wind Stresses in Chimneys.

The oblique lines relate to various percentages of steel from 0 to 2, and may be called "percentage lines." It will be noticed that there are no m' percentage lines for eccentricities less than 0.5; for such small values, the stress f_c' is com-

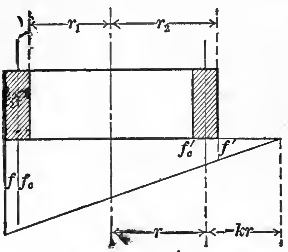


FIG. 3 (a).

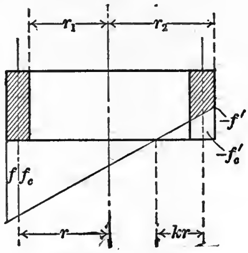


FIG. 3 (b).

pressive (and less than f_c) and there is no need ever to determine these small values of f_c'.

Examples.—(1) A reinforced-concrete chimney is 150 ft. high; its outer diameter is constant and equal to 12 ft. 2 in.; the upper 100 ft. of the shell is 6 in. thick and the lower 50 ft. is 8 in. thick. At a section 50 ft. from the top the vertical reinforcement consists of sixteen $\frac{3}{4}$-in. rods. The extreme unit stresses at this section are required, the chimney being under wind pressure assumed to be equivalent to 30 lbs/ft² of "projected area."

Taking the weight of concrete as 150 lbs/ft³, W is about 137,500 lbs., and since M is about 5,475,000 in-lbs., $e=M/W=39.8$ in., and $e/r=0.57$. Since $A_s=7.07$ in², and $A=2636$ in², $p=0.0027=0.27\%$. With these values of p and e/r, the diagram (Fig. 2) gives $m=2.06$ and $m'=0.135$; hence $f=2.06\ W/A=107$ lbs/in², and $f'=0.135\ W/A=7$ lbs/in². This f' being a small tension, the stress condition at the section under consideration does fall under Case I.

(2) At the section 100 ft. below the top of the chimney the vertical reinforcement consists of forty-eight $\frac{3}{4}$-in. rods. It is required to determine the extreme unit stresses there.

Here W is about 275,000 lbs., and M about 21,900,000 in-lbs.; hence $e=M/W=79.6$ in. and $e/r=1.137$. Since $A_s=21.2$ in², and $A=2636$ in², $p=0.008=0.8\%$. These values of e/r and p fall beyond the limits of the diagram; hence the stress condition does not fall under Case I probably. Substituting in eq. (6), it will be found that $m'=1.15$; and hence $f'=1.15\ W/A=120$ lbs/in² approximately; this being a high tension, the stress condition does not fall under Case I and the methods of Case II should be applied (see ex. 1, Art. 282).

Analysis for the Diagram.—In the case of a uniformly varying stress, the average unit stress for any portion of the section equals the actual unit stress at the centroid of that portion (or at any point of the section whose distance from the neutral axis equals that of the centroid). Hence the average unit stress in the concrete is $\frac{1}{2}(f_c+f_c')$; also the average unit stress in the steel is $\frac{1}{2}(f_c+f_c')n$ (see Fig. 3). (It is supposed that the vertical steel is securely tied to the circumferential so that the former will not buckle.)

And since the total stress on the section equals W,

$$\tfrac{1}{2}(f_c+f_c')A+\tfrac{1}{2}(f_c+f_c')npA=W. \quad . \quad . \quad . \quad . \quad (1)$$

In the case of a uniformly varying stress, the point of application of the resultant of the stress on any portion of the section lies at a distance from the neutral axis equal to the ratio between the square of the radius of gyration of the portion with respect to the neutral axis and the distance of the centroid of that portion from the same axis. Now the radius of gyration of the concrete section is nearly the same as that of the steel circle (radius r), and hence the resultants of the concrete and steel stresses practically coincide. The square of the radius of gyration of this circle with respect to the neutral axis is $\frac{1}{2}r^2+(1-k)^2r^2$ (see Fig. 3); hence the arm of the resultants with respect to the neutral axis is

$$[\tfrac{1}{2}r^2+(1-k)^2r^2]/(1-k)r,$$

and the arm with respect to the center of the section is $r/2(1-k)$. Since the sum of the moments of these two resultants with respect to the center equals the bending moment,

$$\tfrac{1}{2}(f_c+f_c')A\frac{r}{2(1-k)}+n\tfrac{1}{2}(f_c+f_c')pA\frac{r}{2(1-k)}=We. \quad . \quad (2)$$

From eqs. (1) and (2), it follows that

$$k=1-\frac{0.5}{e/r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From the similar triangles in Figs. 3 (*a*) and 3 (*b*), and eq. (3), it follows that

$$f_c' = -f_c k/(2-k) = f_c(1-2e/r)/(1+2e/r),$$

and if this value be substituted in (1), then the equation gives

$$f_c = \frac{1+2e/r}{1+np}\frac{W}{A}; \quad \ldots \ldots \quad (4)$$

hence, also,

$$f_c' = \frac{1-2e/r}{1+np}\frac{W}{A}. \quad \ldots \ldots \quad (5)$$

Since by definition, $f_c = mW/A$ and $f_c' = m'W/A$,

$$m = \frac{1+2e/r}{1+np}, \quad \text{and} \quad m' = \frac{1-2e/r}{1+np}. \quad \ldots \quad (6)$$

The straight line in Fig. 4 was plotted from eq. (3), and all lines in Fig. 2 from eqs. (6).

It should be noticed that f_c and f_c' are not the unit stresses for the extreme fiber; these latter might be obtained from the former and k by proportion (see Figs. 3 (*a*) and 3 (*b*), or by the

Second Method.—This is the ordinary method for combining "direct" and flexural stress; it gives the unit stresses for the extreme fibers. Thus, I denoting the moment of inertia of the concrete-steel section about a diameter, computed as explained below, then

$$f = \frac{W}{A} + \frac{Mr_2}{I},$$

and

$$f' = \frac{W}{A} - \frac{Mr_2}{I}.$$

If, in a given instance, f' comes out negative, then the stress at the windward side of the section under consideration is tensile. The greatest compressive unit stress in the steel is less than nf, and, if some of the steel is under tension, its

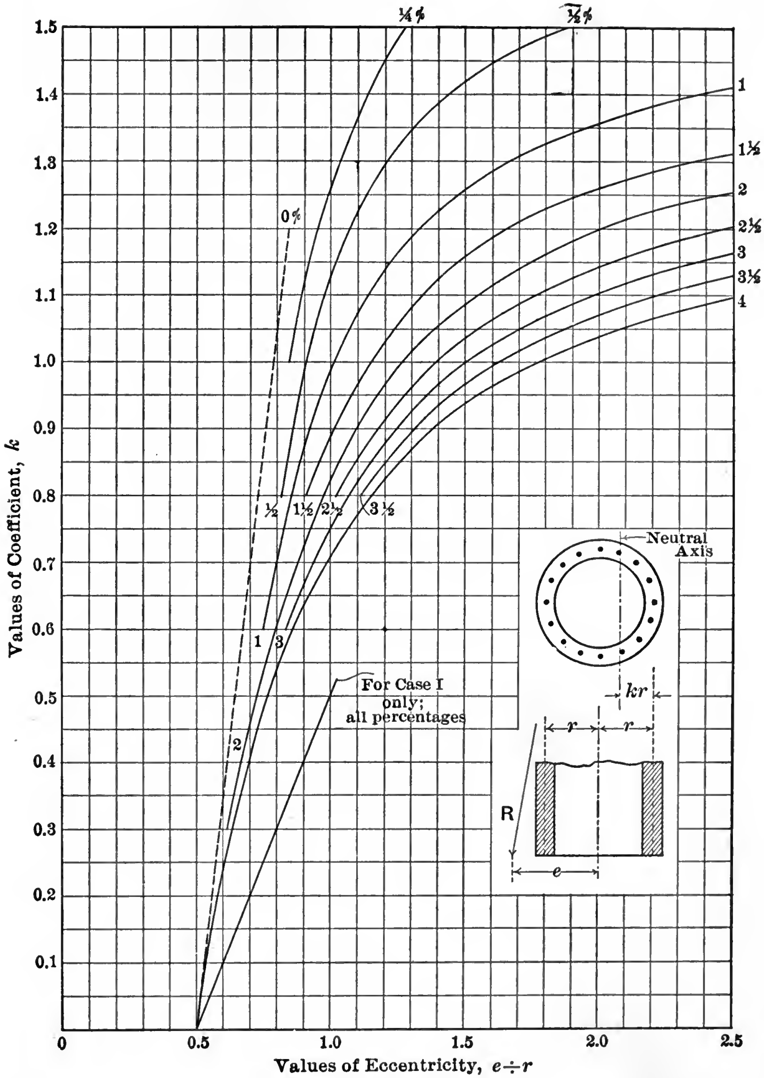


FIG. 4.—Neutral Axes in Chimneys.

greatest tensile unit stress is less than nf'. The first of these maxima is safe if f is safe, and the second is insignificant.

In computing the moment of inertia I, the steel sections must be weighted n-fold; thus let I_c=moment of inertia of the concrete section with respect to a diameter, and I_s=moment of inertia of all the steel sections with respect to the same line, then $I=nI_s+I_c$. Instead of using the actual steel sections to compute I, one may substitute with sufficient accuracy the section of a cylindrical shell rolled from the steel, the mean radius of the shell being r. Then, as the concrete sectional area is practically the same as the total,

$$I=nA_s\tfrac{1}{2}r^2+A\tfrac{1}{4}(r_1^2+r_2^2)=\tfrac{1}{4}A(2npr^2+r_1^2+r_2^2).$$

For Ex. (1), $I=\frac{1}{4}\ 2636(30\times 0.0027\times 70^2+67^2+73^2)=$ 6,731,685 in^4; hence $f=111$ and $f'=-7$ lbs/in^2, the negative sign indicating tensile stress at the windward side. For Ex. (2), $I=6{,}970{,}000$ in^4, $f=334$ and $f'=-126$ lbs/in^2.

282. *Case II.*—The eccentricity is so great that the resultant stress at the windward side is a tension whose intensity is so high that the concrete has been cracked or is near the cracking stage; in other words, the tensile stress condition resembles somewhat that at the section of maximum moment in an ordinary reinforced-concrete beam under full safe load. In the computation for this case the tensile strength of the concrete will be entirely neglected, as is almost universally done nowadays for concrete beams.

Practical formulas for unit stresses based on this "common theory" cannot be deduced for this case. But a diagram can be constructed by means of which unit stresses can be easily determined for any section of a given chimney; also the amount of vertical reinforcement required at any section of a given concrete shell can be readily determined by it. Such a diagram will now be described by example, and then the analysis on which its construction is based will be given. At the base (Fig. 5) there are given values of "eccentricity"

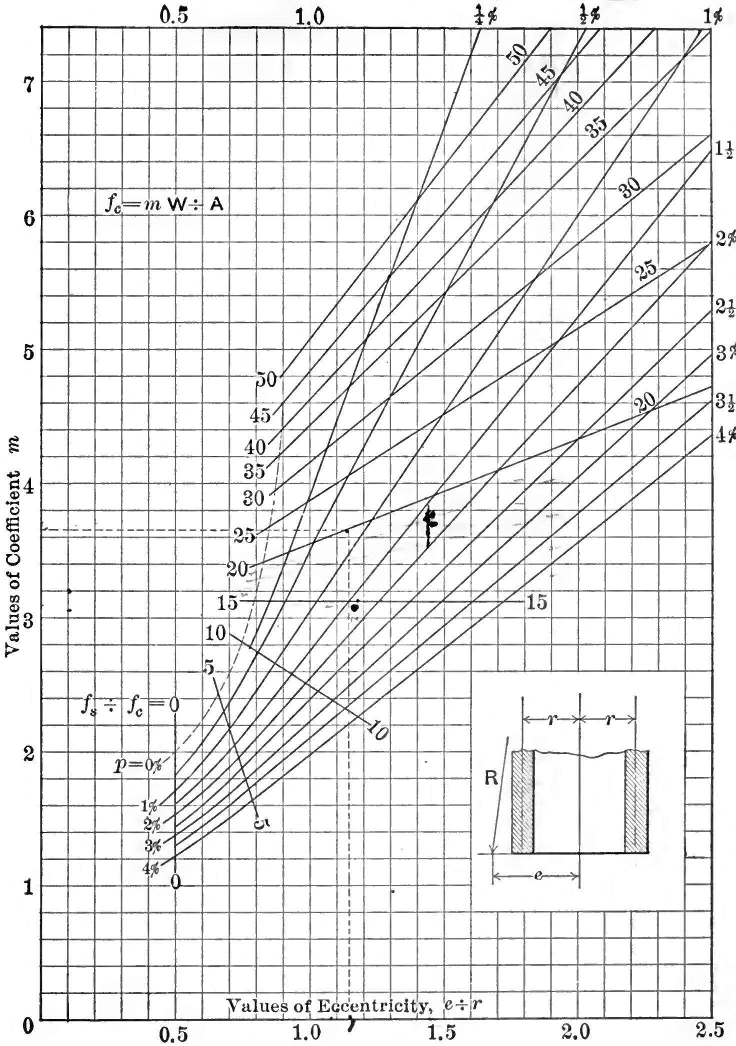


FIG. 5.—Wind Stresses in Chimneys

e/r, and at the left side values of m. The curves relate to various percentages of steel, from 0 to 4%, and will be called percentage lines; and the straight lines relate to various ratios of f_s/f_c and will be called ratio lines.

Examples.—(1) Ex. (2), Art. 281, will be used as illustration. From its solution, $W=275{,}000$ lbs., $M=21{,}900{,}000$ in-lbs., $p=0.8\%$, and $A=2{,}636$ in²; hence $e=21{,}900{,}000/275{,}000=79.6$ in., and $e/r=79.6/70$ $=1.14$; also $W/A=104$ lbs/in². Now entering the diagram at $e/r=1.14$, we trace vertically upward to a point corresponding to an 0.8 percentage-line, and then horizontally to the left side, taking out the value $m=3.65$. We also note that the turning point is practically at the 20 ratio-line. Hence $f_c=3.65\times104=380$ lbs/in², and $f_s=20\times380=7600$ lbs/in². It may be noticed that f_c is not the unit stress at the remotest fiber, and hence not the maximum compressive unit stress in the section. The maximum can be readily computed from

$$f=f_c+(f_c+f_s/n)t/4r,$$

in which t denotes thickness of the concrete shell. (The formula may be deduced from similar triangles in the lower part of Fig. 6.) Here

$$f=380+(380+500)\ 6/280=399 \text{ lbs/in}^2.$$

(2) How much vertical reinforcement is needed at the base of the chimney, the working strengths of concrete and steel being limited to 500 and 15,000 lbs/in² respectively?

W is about 456,000 lbs., the wind pressure about 54,750 lbs., and M about 4,106,000 ft-lbs.; hence $e=4{,}106{,}000/456{,}000=9$ ft. $=108$ in., and $e/r=108/69=1.56$. The section area A is 3490 in², and W/A $=130$ lbs/in²; hence if the amount of steel is just sufficient to make $f_c=500$, then $m=500/130=3.85$. Now entering the diagram at $e/r=$ 1.56 and $m=3.85$, we trace vertically and horizontally from these places respectively, and note the intersection at about $p=1.9\%$ and $f_s/f_c=19$. With this percentage of steel, $f_s=19\times500=9500$ lbs/in². (This is a low working stress; use of a thicker shell will make higher values possible without increase of amount of steel. Several trial sections with different thicknesses may be quickly analyzed by means of the diagram, and an economical size determined.)

Analysis for the Diagram.—The ring $NPNQ$ (Fig. 6) represents a section of a chimney, NN the neutral axis, and

NQN the compression area, the wind blowing from the right. In addition to the foregoing notation, let

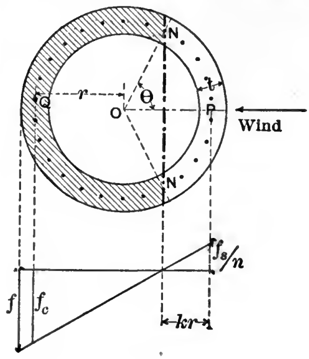


FIG. 6.

C_c = resultant compresive stress in the concrete;

C_s = resultant compressive stress in the steel;

T = resultant tensile stress in the steel;

a_c = arm of the resultant compression C_c+C_s with respect to the center O; and

a_t = arm of the resultant tension T with respect to the center O.

The resultant normal stress on the section equals the bending moment, that is,

$$C_c+C-T=W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$(C_c+C_s)a_c+Ta_t=M. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two equations constitute the basis of the solution; however, they must be modified considerably, and this will be done presently. Let

x_1 = distance from NN to the centroid of the arc NPN;
x_2 = distance from NN to the centroid of the arc NQN;
y_1 = radius of gyration with respect to NN of arc NPN;
y_2 = radius of gyration with respect to NN of arc NQN; and
θ = angle NOP.

Now the average unit compressive stress in the concrete is $f_c x_2/(1+\cos\theta)r$, that in the compressive steel is $nf_c x_2/(1+\cos\theta)r$, and that in the tensile steel is $nf_c x_1/(1+\cos\theta)r$. And since the area of the section of the compressive concrete is practically $A(1-\theta/\pi)$, that of the section of the compressive steel $pA(1-\theta/\pi)$, and that of the tensile steel $pA\theta/\pi$, it follows that

$$C_c=A(1-\theta/\pi)f_c x_2/(1+\cos\theta)r,$$

$$C_s=pA(1-\theta/\pi)nf_c x_2/(1+\cos\theta)r,$$

and $$T = pA(\theta/\pi)nf_c x_1/(1+\cos\theta)r.$$

These values of C_c, C_s, and T substituted in eq. (1) give

$$(1-\theta/\pi)(1+np)x_2/r-(\theta/\pi)np\,x_1/r=(1+\cos\theta)W/Af_c. \quad (3)$$

From any source of information on the centroid of a circular arc, it can be shown that

$$x_1=r\left(\frac{\sin\theta}{\theta}-\cos\theta\right) \quad \text{and} \quad x_2=r\left(\frac{\sin\theta}{\pi-\theta}+\cos\theta\right).$$

Imagining these values substituted in eq. (3), it will be seen that the left-hand member is a function of θ, n, and p only. Denoting this function by $F_1(\theta, n, p)$, the equation can be written thus,

$$f_c=\frac{1+\cos\theta}{F_1(\theta, n, p)}\frac{W}{A}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

hence, also,

$$m=\frac{1+\cos\theta}{F_1(\theta, n, p)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

that is, m depends on θ, n, and p only.

Referring to statement in the preceding article about the point of application of resultant stress and to Fig. 6, it will be seen that

$$a_t=y_1^2/x_1+r\cos\theta.$$

And since the resultants C_c and C_s are practically equally distant from the neutral axis, the arms of C_c+C_s and C_s are practically equal; that is, approximately,

$$a_c=y_2^2/x_2-r\cos\theta.$$

If now these values of a_t and a_c and those for C_c+C_s and T be substituted in eq. (2), it will reduce to

$$\left(1-\frac{\theta}{\pi}\right)(1+np)\frac{x_2}{r}\left(\frac{y_2^2}{x_2r}-\cos\theta\right)+\frac{\theta}{\pi}np\frac{x_1}{r}\left(\frac{y_1^2}{x_1r}+\cos\theta\right)$$
$$=(1+\cos\theta)\frac{W}{Af_c}\frac{e}{r}. \quad . \quad . \quad . \quad . \quad (6)$$

It can be shown that $y_1^2=r^2[1+\frac{1}{2}\cos 2\theta-\frac{3}{4}(\sin 2\theta)/\theta]$ and $y_2^2=r^2[1+\frac{1}{2}\cos 2\theta+\frac{3}{4}(\sin 2\theta)/(\pi-\theta)]$. Imagining these values of y_1 and y_2 substituted in eq. (6), it will be seen that the

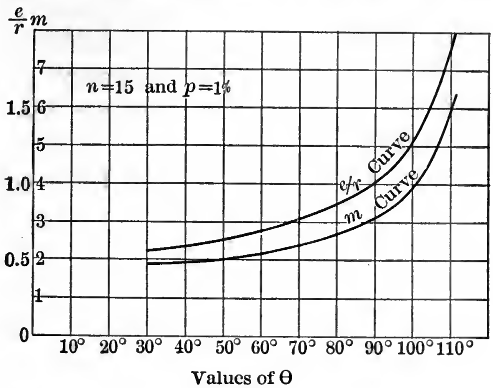


FIG. 7.

left-hand member is a function of θ, n, and p only. Denoting this function by $F_2(\theta, n, p)$, the equation can be written,

$$(1+\cos\theta)(e/r)W/Af_c=F_2(\theta, n, p). \quad . \quad . \quad . \quad (7)$$

Division of eq. (7) by eq. (4) gives

$$\frac{e}{r}=\frac{F_2(\theta, n, p)}{F_1(\theta, n, p)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

that is, e/r depends on θ, n, and p only.

Equations (5) and (8) are the desired modifications of eqs. (1) and (2). If both be plotted on a θ base for a given set of values of n and p ($n=15$ and $p=0.01$, say), a pair of curves results as sketched in Fig. 7, from which may be taken the value of m for any value of e/r. Finally, if such simultaneous values of m and e/r (corresponding to one and the same value of θ) be taken off from this pair of curves and these values be plotted on an e/r base, the resulting curve is the 1% line of the diagram, Fig. 5. In a similar manner the other percentage lines can be determined.

The ratio lines may be determined as follows: From similar triangles in the lower part of Fig. 6 it is plain that

$$f_s/nf_c=(1-\cos\theta)/(1+\cos\theta), \quad \text{or} \quad f_s/f_c=n\tan^2\theta/2.$$

From this equation it appears that for $f_s/f_c=5$, say, and $n=15$, $\theta=60°$, irrespective of the value of p. Now the values of e/r for $\theta=60°$, and $p=0, \frac{1}{2}, 1, 1\frac{1}{2}$, etc., may be read off from the corresponding pairs of curves (like Fig. 7), and these values of e/r may be marked off on the corresponding percentage curves in the diagram; the points so marked off fix the $f_s/f_c=5$ line. In a similar manner the other ratio lines can be determined.

Since $kr=r-r\cos\theta$ (see Fig. 6), $k=1\,(1-\cos\theta)$. From this formula and Fig. 7 the value of k may be obtained for any value of e/r and $p=1\%$. In this way the 1% curve in Fig. 4 was obtained; and in a similar way the others.

283. **Wind Pressure.**—Recent experiments made on the Eiffel Tower and at the National Physical Laboratory of England show that the pressure in lb/ft^2 on square flat surfaces from 10 to 100 square feet in extent is 0.0032 times the square of the wind velocity in miles per hour. There is some evidence that the pressure on a cylindrical surface is about two-thirds that which would exist on an axial section of the cylinder ("projected area"). On the basis of the above, 20 pounds per square foot of projected area is a safe value for chimneys. The Prussian Regulations permit use of 17 pounds per square foot; in American practice considerably higher values are used.

284. **Temperature Stresses.**—Fig. 8, which is from a photograph, shows plainly some large vertical and horizontal cracks in the outer shell of a chimney. The vertical cracks are doubtless due to temperature and probably the horizontal ones also. For since the inner part of the shell is hotter than the outer, the inner tends to expand more circumferentially and vertically than the outer, so that it stretches the outer part and is itself compressed vertically and circumferentially by the outer part. If the circumferential or vertical tensions in the outer part are excessive, the concrete will crack on vertical or horizontal planes respectively.

285. *Circumferential Temperature Stress.*—The following are formulas for the greatest unit compression in the con-

crete and the unit tension in the steel at any place in a chimney:

$$f_c = \tau K E_c m_c \quad \text{and} \quad f_s = \tau K E_s m_s,$$

m_c and m_s being certain multipliers which depend on the percentage of hoop reinforcement, the position of the hoops

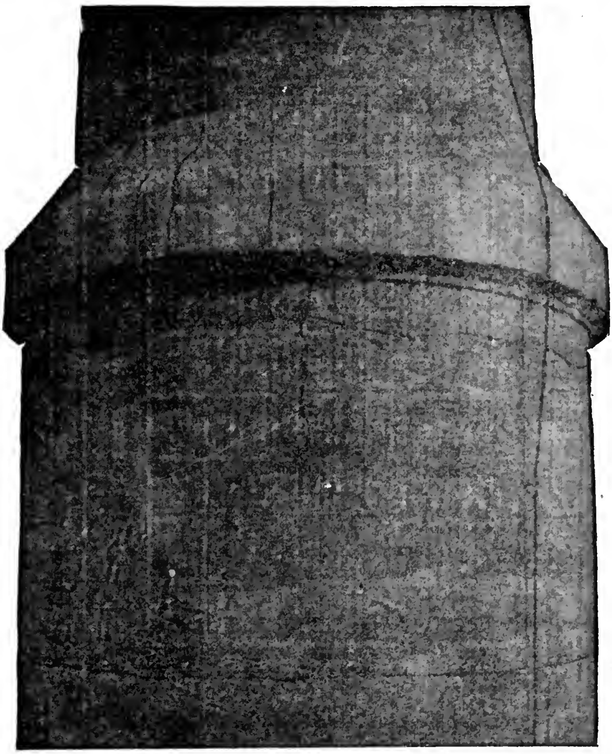
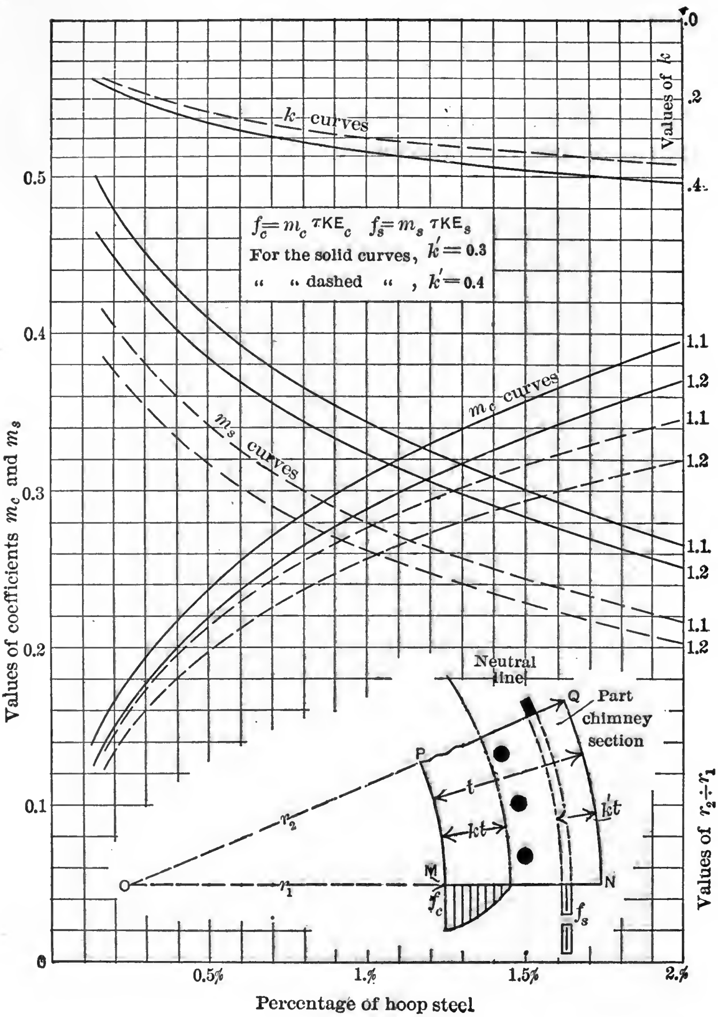


Fig. 8.

relative to outer and inner surfaces of the shell, and the ratio of outer to inner radii. For meaning of the other symbols see page 427. Formulas for the multipliers are complicated (page 429) but can be made available for practical use by graphical means. They have been plotted in a diagram (Fig.



k curves
Values of k
.0
.2
.4
0.5
$f_c = m_c \tau KE_c$ $f_s = m_s \tau KE_s$
For the solid curves, $k' = 0.3$
" " dashed " , $k' = 0.4$
0.4
0.3
0.2
0.1
0
m_c curves
m_s curves
1.1
1.2
1.1
1.2
1.1
1.2
1.1
1.2
Values of coefficients m_c and m_s
Neutral line
Q
Part chimney section
P
t
kt
k't
r_2
r_1
O
M
N
f_c
f_s
Values of $r_2 \div r_1$
0.5%
1.%
1.5%
2.%
Percentage of hoop steel

Fig. 9.

9) for percentages of steel from 0.2 to 2, for proportionate depths of steel from 0.3 to 0.4, and for ratios of outer to inner radii from 1.1 to 1.2. Inspection of the diagram shows:

(1) The higher the percentage of steel, the lower is the unit stress in the steel f_s and the higher is that in the concrete f_c.

(2) Increasing the thickness of the shell within practical limits decreases the unit stresses in concrete and steel but not materially, less than 10%.

(3) Moving the steel inward from $0.3t$ to $0.4t$ decreases both unit stresses, from 10 to 20% depending upon the amount of reinforcement.

Example.—The internal diameter of a chimney shell is 12 ft., the thickness of its walls is 6 in., the hoops are $\frac{1}{2}$-in. rounds 10 in. apart, and are placed so that the center of the steel is 2 in. from the outer surface of the shell. What are the temperature stresses in steel and concrete due to a temperature difference of 200° F.?

The percentage of steel is $0.1964/(6\times10)=0.0033=1/3\%$, $r_2/r_1=1.1$, and $k'=2/6=1/3$. For $k'=0.3$, $p=1\%$ and $r_2/r_1=1.1$, the diagram gives $m_c=0.20$ and $m_s=0.445$; for $k'=0.4$, $m_c=0.18$, and $m_s=0.375$. Hence by interpolation for $k'=1/3$, $m_c=0.19$ and $m_s=0.42$. With $K=0.000006$, $E_c=2{,}000{,}000$ and $E_s=30{,}000{,}000$ lbs./in.^{2},

$$f_c=200\times0.000006\times2{,}000{,}000\times0.19=456$$

and
$$f_s=200\times0.000006\times30{,}000{,}000\times142=15{,}100 \text{ lbs./in.}^2$$

Analysis for the Diagram.—In Fig. 9, $MNPQ$ represents a portion of a horizontal section of a chimney and O the center of the section. The "neutral line" represents the neutral (cylindrical) surface which is not stressed, within and without which the concrete is under compression and tension respectively. The distance of the circumferential steel from the outer surface of the shell is called $k't$; r denotes the radius of any circumferential "fiber," C the total compressive stress on the vertical section MN per foot of height, and T the total tension in the circumferential steel per foot of height. Also let

t = thickness of concrete shell at the section under consideration;

r_1 = inner radius of shell;

r_2 = outer radius of shell;

τ = difference of temperatures of concrete at outer and inner faces;

K = coefficient of expansion of concrete and steel;

E_c = modulus of elasticity of concrete in compression;

E_s = modulus for steel;

f_c = temperature unit stress in concrete (circumferential) at inner face; and

f_s = temperature unit stress in circumferential steel.

In this analysis the tensile value of the concrete is neglected and an average modulus of elasticity for concrete in compression is assumed for all unit stresses, just as in computations on the strength of reinforced-concrete beams. The temperature gradient is assumed to be straight and the coefficients of linear expansion for concrete and steel are taken as equal and constant for all temperatures involved (see Art. 42). Furthermore, it is assumed that the thickness of the shell remains unchanged and that the radii of all circumferential fibers are increased equally. (Strictly this is not true for there are radial expansions and contractions accompanying the circumferential stresses, and then there is radial shortening accompanying the radial compressive stress; but these are small and their observance is out of place in this analysis involving as it does, approximation of a larger order. On account of the unequal vertical expansions there will be circumferential expansion at the top, the chimney "belling" out there, and some contraction lower down. In the following analysis, these are neglected; the resulting error is probably small except for stresses near the top. The assumption that the modulus of elasticity is constant for the range in temperature may be quite erroneous, and if so, the analysis following is in error; if the modulus is lower for the higher

temperatures then the actual unit stresses are lower than those here found.)

If Δr denotes the radial increase mentioned, then the actual elongation of each circumference is $2\pi\Delta r$, and the actual unit elongation is $\Delta r/r$. The temperature at the distance r from the center (reckoned from the temperature at the outer surface as zero) is $\tau(r_2-r)/t$, and hence the free elongation of the circle of radius r would be $2\pi r K\tau(r_2-r)/t$ and the free unit elongation $K\tau(r_2-r)/t$. The difference between the free and the actual unit elongations is the prevented unit elongation, and hence the corresponding preventing unit stress in the concrete is

$$f=\left[\frac{\tau}{t}(r_2-r)K-\frac{\Delta r}{r}\right]E_c; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the steel, r becomes equal to $r_2-k't$ and

$$f_s=\left[\frac{\Delta r}{r_2-k't}-\tau k'K\right]E_s. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

At the neutral surface $f=0$, and $r=r_1+kt$; hence on substituting these in eq. (1), it will be found that

$$\Delta r=\tau K(1-k)(r_1+kt). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A value of the compression at the inner face may be obtained from (1) by substituting for r its value there and for Δr its value from eq. (3); the expression will reduce to

$$f_c=\tau KE_c[1-(1-k)t/r_1]k. \quad . \quad . \quad . \quad . \quad . \quad (1)'$$

A new value of f_s can be obtained from (2) by substituting for Δr its value from eq. (3); it will reduce to

$$f_s=\tau KE_s\left[\frac{(1-k)(1+kt/r_1)}{(r_2/r_1)-k't/r_1}-k'\right]. \quad . \quad . \quad . \quad (2)'$$

These expressions for f_c and f_s contain only quantities ordinarily known and k; it remains to determine k. This is done by means of the condition that C and T are equal. Now,

$C=\int_{r_1}^{r_1+kt} f\,12dr$ and $T=f_s p\,12t$; and if in these, the values of f and f_s from eqs. (1) and (2) be inserted, and the resulting expressions for C and T be equated, and then if the operations indicated in the equation be performed, the following may be arrived at:

$$\Delta r=K\tau t(k-\tfrac{1}{2}k^2+npk')(r_2/t-k')\div[np+(r_2/t-k')\log(1+k't/r_1)]$$

The logarithm is Naperian, or natural. Equating this value of Δr to that given by eq. (3) and simplifying, one may arrive at

$$\frac{k-\frac{1}{2}k^2+npk'}{(1-k)(k+r_1/t)}=\log(1+kt/r_1)+\frac{np}{r_2/t-k'}. \quad . \quad . \quad (5)$$

Now this equation determines k for given values of k', n, and r_2/r_1, and yet it cannot be solved for k on account of the logarithmic term. However, values of p can be determined for given values of k, k', n, and r_2/r_1, and thus a diagram can be made like the group of k curves (Fig. 9) but with much larger vertical scale, from which k may be taken off for any given case (p, k', n, r_2/r_1). Then this value may be used in eqs. (1)′ and (2)′ to determine the bracketed coefficients, that is, m_c and m_s.

286. *Vertical Temperature Stress.*—A satisfactory analysis is not available; the following approximation indicates the "order of magnitude" of these stresses. Horizontal sections through the unheated chimney are still horizontal after heating just as plane sections of a beam remain plane during bending. Hence the inner part of the chimney shell when hot is compressed while the outer part is stretched, and somewhere between there will be a neutral surface whose distance from the inner surface is here called kt (see Fig. 10). If the tensile strength of the concrete is neglected, then the entire vertical tension must be ascribed to the steel, and the neutral surface located between the steel and inner surface as shown.

The temperature difference between the inner surface and the neutral surface is τk, and the temperature difference between the neutral surface and the steel is $\tau(\frac{1}{2}-k)$; hence the unit stress at the inner surface f_c, and the unit stress in the steel f_s are given by

$$f_c = k\tau KE_c, \qquad (1)$$

and

$$f_s = (\tfrac{1}{2}-k)\tau KE_s. \qquad (2)$$

To determine k equate the total compression and the total tension per unit of circumference; thus p denoting the (vertical) "steel ratio," or total vertical steel area divided by

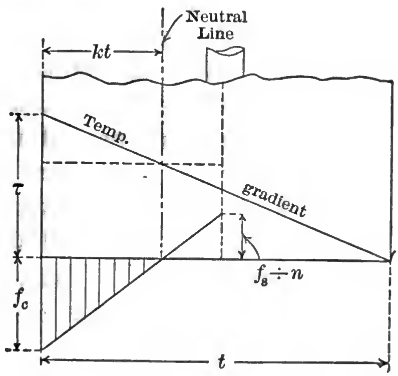


FIG. 10.

total area of cross-section of the shell, then $\frac{1}{2}k\tau KE_c kt = (\frac{1}{2}-k)\tau KE_s pt$, which simplified and solved for k gives

$$k = np[\sqrt{(1+1/np)} - 1]. \qquad (3)$$

From this equation the following table was computed, n taken as 15:

$p=$	.005	.010	.015	.020	.025	.030	.035	.040
$k=$	0.21	0.26	0.30	0.32	0.34	0.36	0.37	0.38

For 2% of steel, $\tau = 200°$, and $K = 0.000006$, formulas (1) and (2) give $f_c = 770$ and $f_s = 6500$ lbs/in^2.

287. Chimney Temperatures.—Thompson, in the report alluded to in Art. 279, states that "the temperature in an ordinary chimney seldom exceeds 700° F. at the base and 400 to 500 is more usual."

Whatever the difference between the temperatures of the chimney gas and the outer air, the difference τ between the temperatures of the concrete at the inner and outer face of the chimney shell (on which the temperature stresses depend) is less, for it is known that at the surfaces of a heat barrier there is a drop in temperature in the direction of the heat flow. And at the outer surface of a chimney the drop is considerable, as is known to any one who has placed his hand upon the surface on a cold day; to him it felt warm while the air temperature may have been zero or less. Lange* has computed the temperature drops at the surfaces of some brick chimneys (various diameters, thickness of walls, gas, and air temperatures) and while the computations seem to be based on uncertain values of thermal conductivity, emissivity, and absorption of the chimney material, still they are doubtless reliable enough to indicate that the temperature difference for a concrete shell may be as little as 50% of the temperature difference for the chimney gases and air.

288. Bases.—In the two succeding articles there are explained methods for computing the maximum pressures between the base of a given chimney and its earth or stone foundation, and the strength of the base to withstand those pressures and the forces on its top. The notation is as follows (see also Figs. 11 and 14):):

W = total weight of chimney and earth filling over the base;
M = wind moment at the bottom of the base;
A = area of the bottom of the base;
p_2 = maximum unit pressure on bottom;
p_1 = minimum unit pressure on bottom;
r = kern radius of bottom in direction of wind;
e = eccentricity at bottom of resultant of the wind pressure and W, $e = M/W$.

* Der Schornsteinbau.

289. *Earth Pressures.*—It is assumed that the pressure is a uniformly varying one. If the resultant of the wind pressure on the chimney, the weight of the chimney, and earth filling over the base cuts the kern of the bottom of the base, then there will be no tendency for the windward toe to lift, or, in other words, there will be no tendency to tension between the earth or rock foundation and the windward toe. Fig. 11 shows the kerns for square, octagonal, and circular bottoms. It is good practice to make the bottom, in a given case, large enough so that its kern will intercept the line of action of

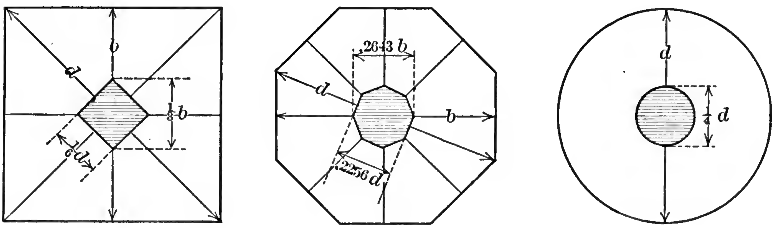


Fig. 11.

the resultant R; however, this is not an absolutely essential requisite for stability and safety.

If the resultant R cuts the kern then the computation for the greatest and least unit pressure is comparatively simple. These are the formulas:

$$p_1=\frac{W}{A}+\frac{M}{Ar} \quad \text{and} \quad p_2=\frac{W}{A}-\frac{M}{Ar}. \quad . \quad . \quad . \quad (1)$$

Evidently the greater pressure p_1 is a maximum when the kern radius is minimum; hence p_1 is maximum when the wind pressure is parallel to the longest diameter of the base.

If the resultant R does not cut the kern then there is a neutral axis as it were, that is, only a part of the bottom is under pressure. This neutral axis is perpendicular to the direction of the wind pressure. Three cases are noted:

(*a*) *Square Bases.*—If the direction of the wind is parallel

to a side of the square, then the distance x of the neutral axis from the windward of the square is

$$x=3(\tfrac{1}{2}b-e), \quad \ldots\ldots\ldots \quad (1)$$

and the greatest unit pressure is

$$p_1=\frac{4}{3(1-2e/b)}\frac{W}{b^2}. \quad \ldots\ldots\ldots \quad (2)$$

These formulas follow from two equations obtained by equating the total pressure on the bottom to W, and the moment of that pressure about the diameter parallel to the neutral axis to $M=We$.

If the direction of the wind is parallel to a long diameter of the square, then the position of the neutral axis (see Fig. 12) is given by

$$\frac{1-\frac{1}{2}k^3(2-k)}{6(1-k)+k^3}=\frac{e}{r}, \quad \ldots\ldots\ldots \quad (3)$$

and the value of the greatest unit pressure by

$$p_1=\frac{2-k}{1-k+\frac{1}{6}k^3}\frac{W}{b^2}=m\frac{W}{b^2}. \quad \ldots\ldots \quad (4)$$

In Fig. 12 there are two curves marked "square"; one gives values of k and the other values of m for use in $p_1=mW/b^2$. Eqs. (3) and (4) were deduced from the same principles employed to deduce (1) and (2).

(b) *Octagonal Bases.*—Exact formulas are not practical. Since an octagon does not differ much from a co-centric circle whose diameter equals the mean of the greatest and least diameters of the octagon, it must be that the neutral axis and the greatest unit pressure for an octagonal bottom do not differ materially from those for such

(c) *Circular Bases.*—In Fig. 12 there are two curves marked "circle"; one of these gives values of k and the other values of m for use in $p_1=mW/A$, A denoting area of the circle and p_1 the greatest unit pressure on it.

290. *Design of Bases.*—Having determined the diameter of the base from a consideration of the earth pressures as explained

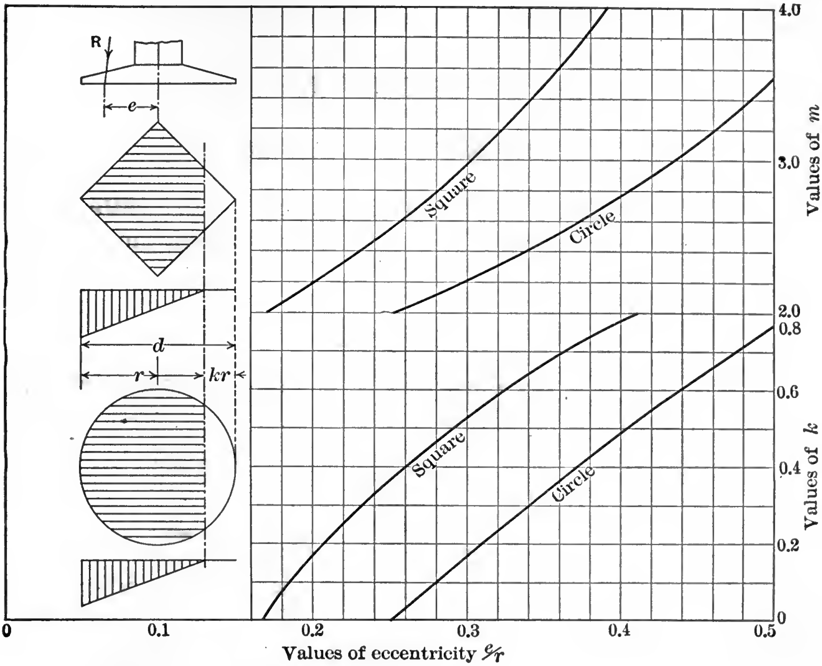


FIG. 12.

in the preceding article, it remains to determine thicknesses of the bases and the reinforcement. Only rough approximate methods are available. A chimney base is essentially a column footing for methods of the design of which see Art. 196. But while column footings are regarded as always subjected to a uniform earth pressure, a chimney base should not be so regarded; and the outstanding cantilever part of a base should be figured for the earth pressure distribution obtaining with the maximum wind pressure as explained in the preceding article. Also while the column is solid, the chimney is not; and in the latter case tensile stresses may arise at the top of the base at its center. Such tension will probably obtain when the inner diameter of the chimney is greater than twice the length of the outstanding cantilever and no wind blowing.

CHAPTER XIV.

DIAGRAMS AND TABLES.

Following is a list of diagrams and tables, together with references to the particular articles in which their construction and use is explained.

DIAGRAMS.

TABLES.

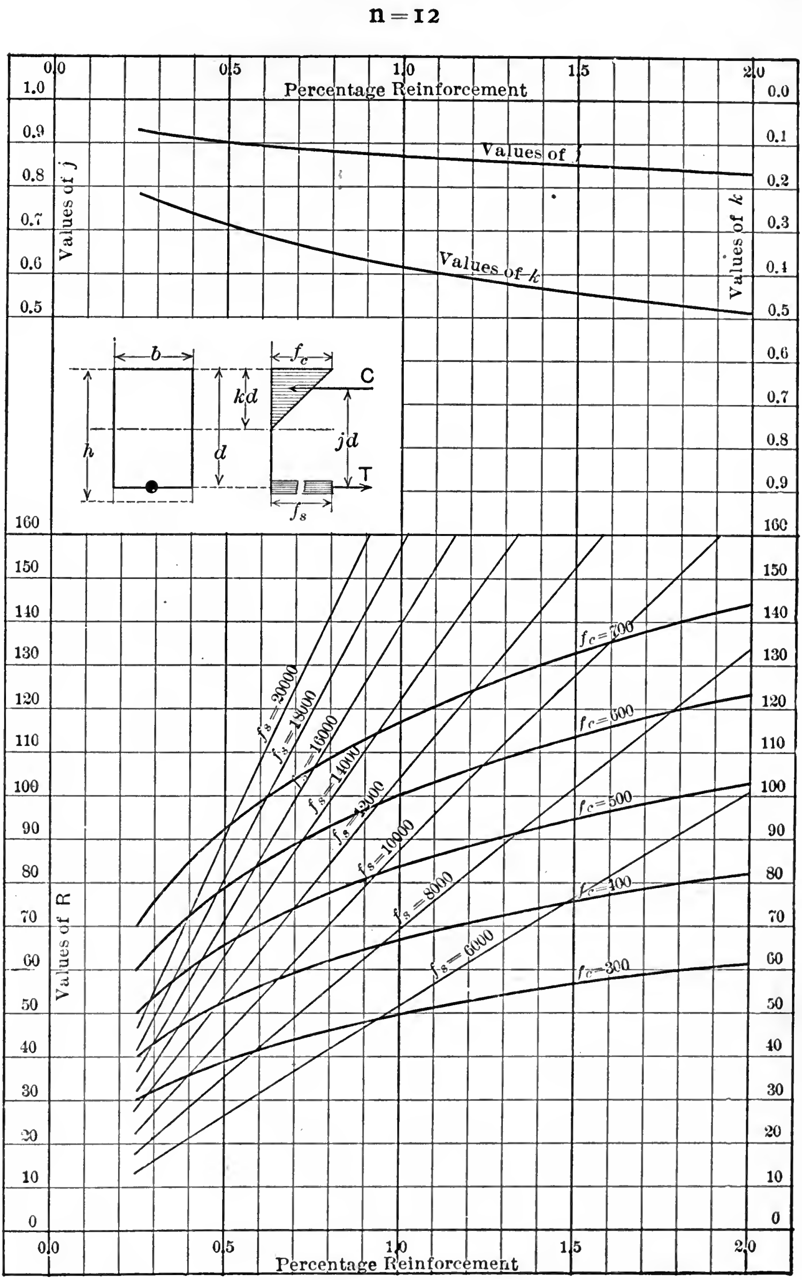


PLATE I.—Coefficients of Resistance of Beams.

n = 15

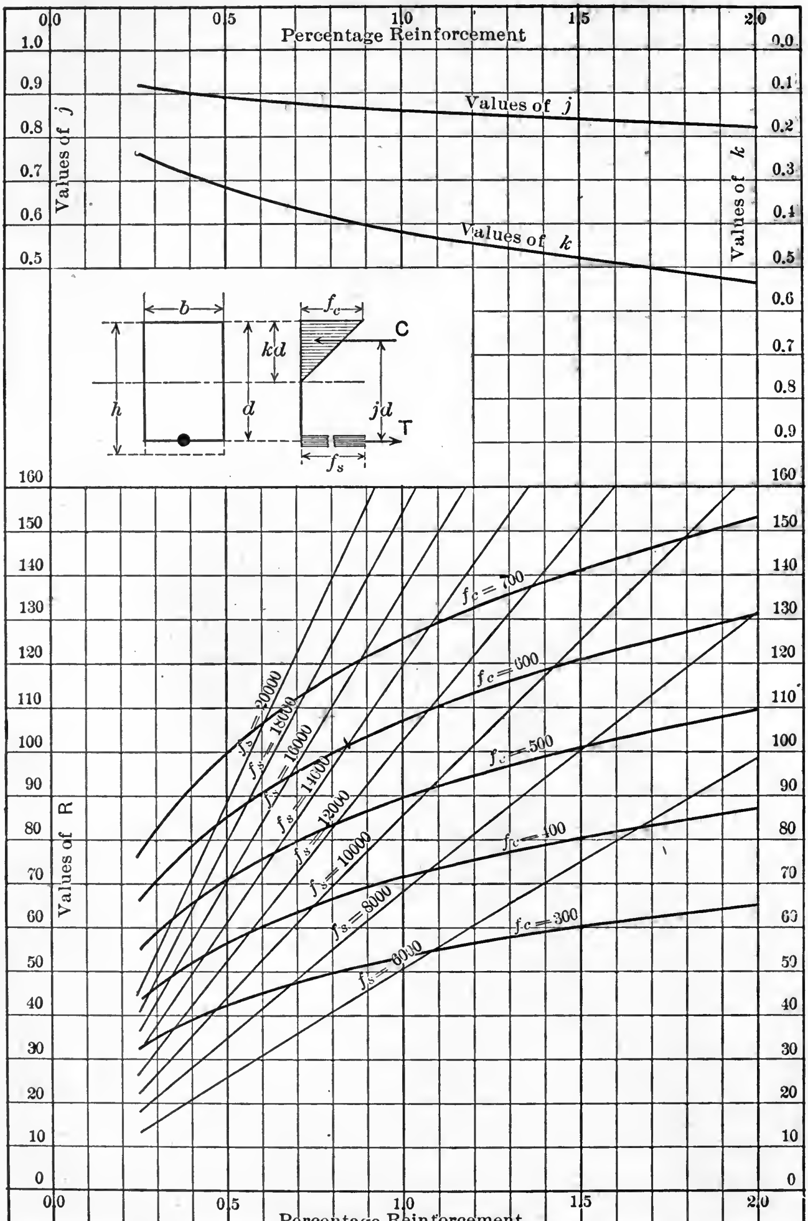


PLATE II.—Coefficients of Resistance of Beams.

Full lines for n = 15; dotted lines for n = 12.

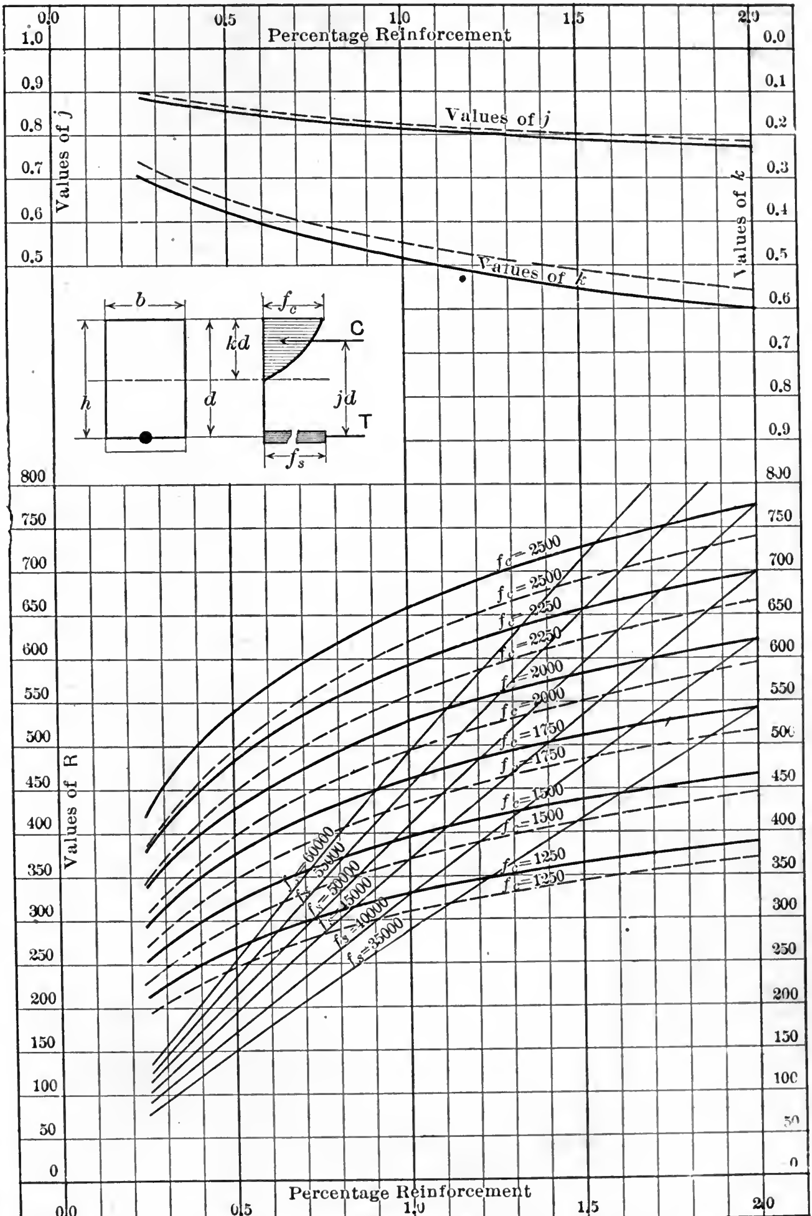


PLATE III.—Coefficients of Resistance of Beams.

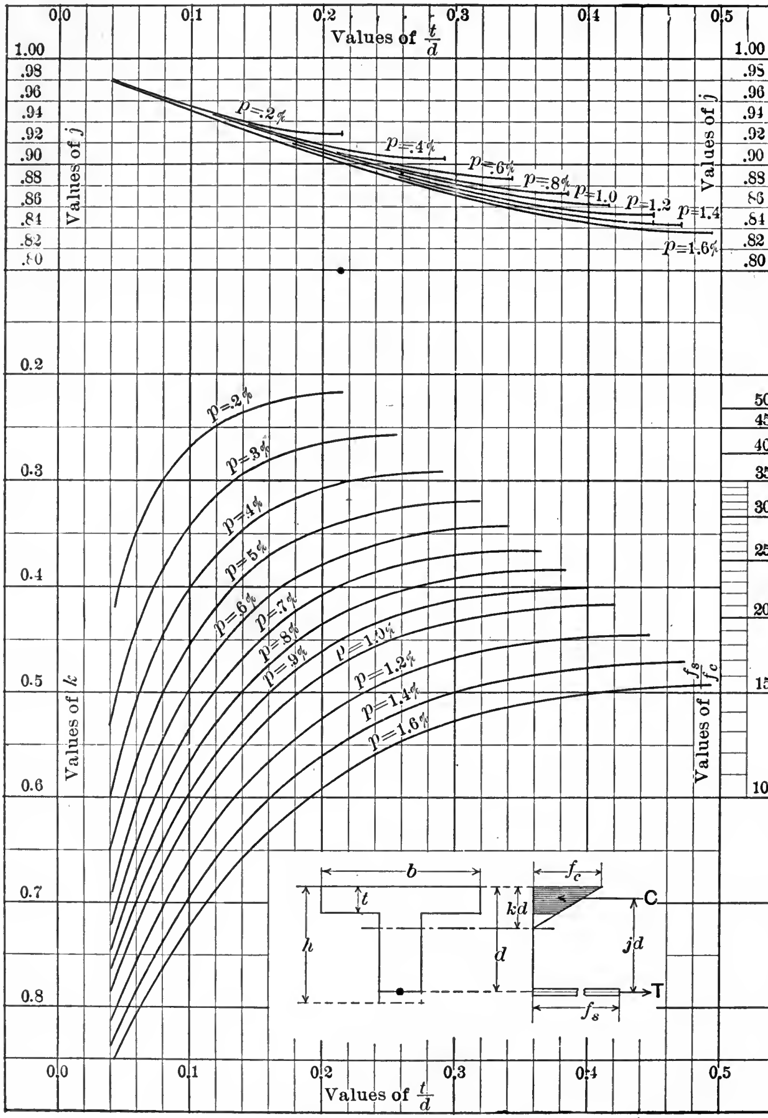


Plate IV.—Values of k and j for T-beams.

$f_s = 14{,}000$

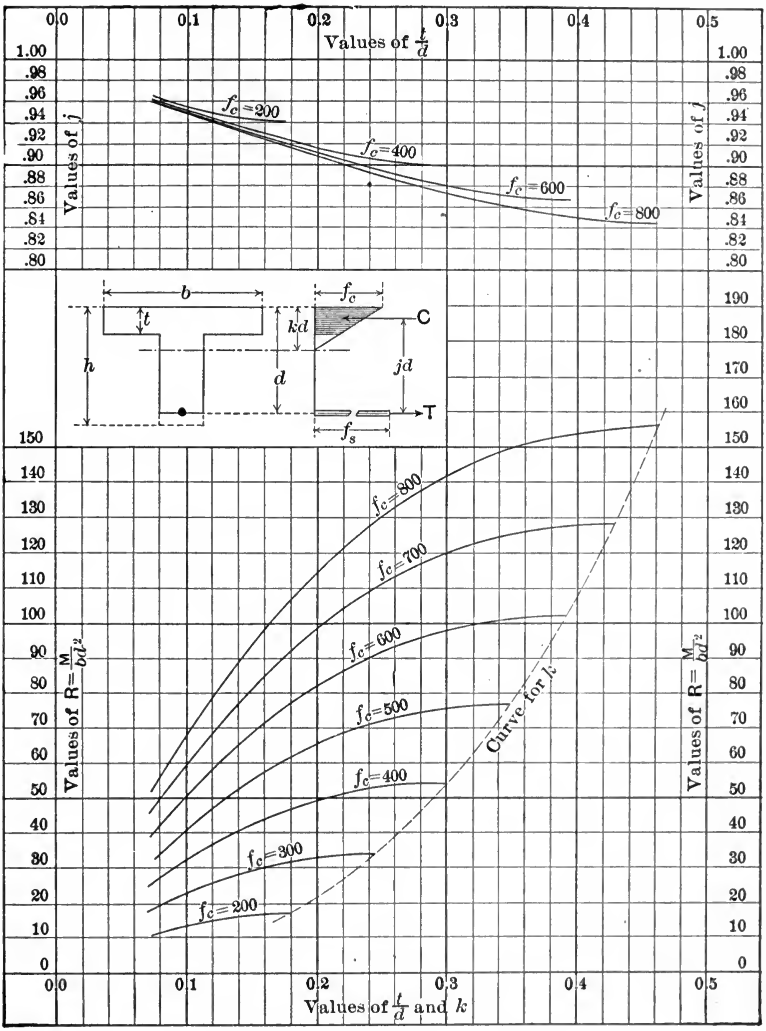


Plate V.—Coefficients of Resistance of T-beams

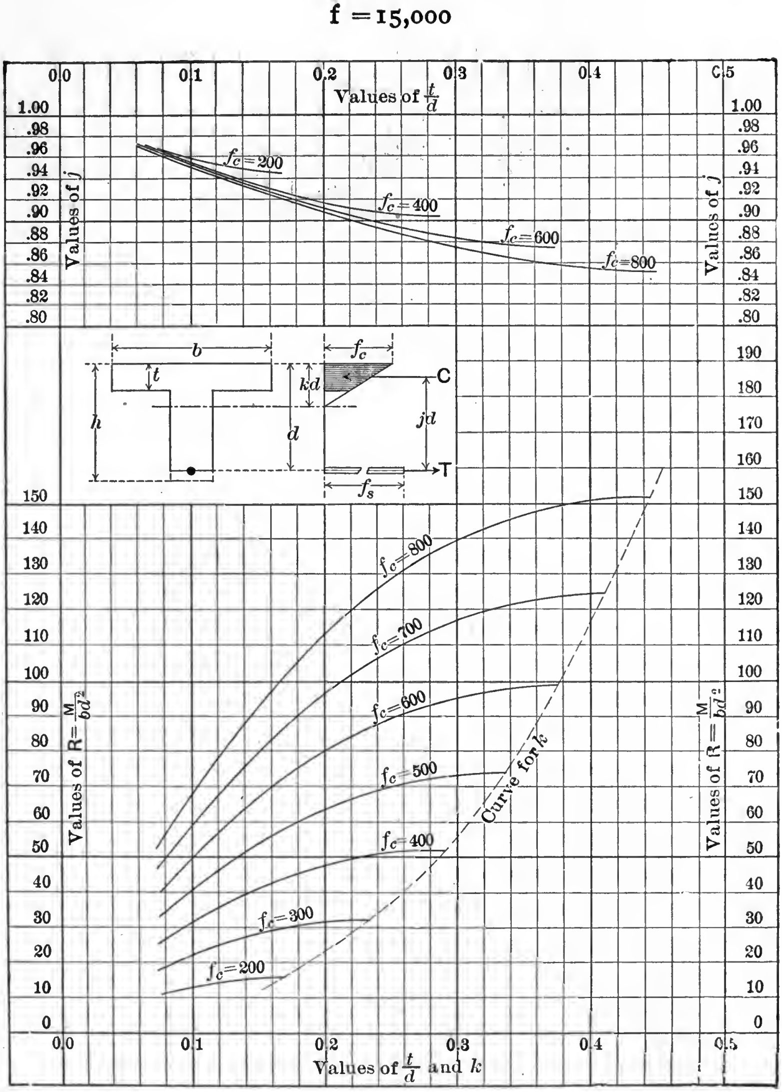


PLATE VI.—Coefficients of Resistance of T-beams.

$f_s = 16{,}000$

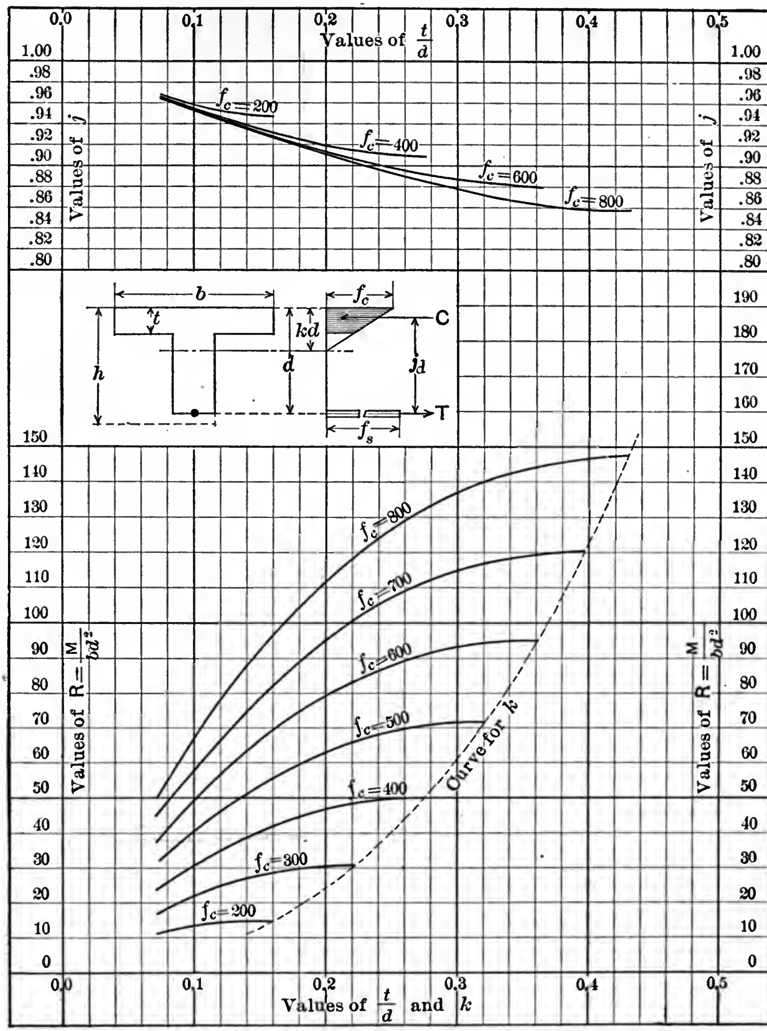


PLATE VII.—Coefficients of Resistance of T-beams.

$f_s = 18{,}000$

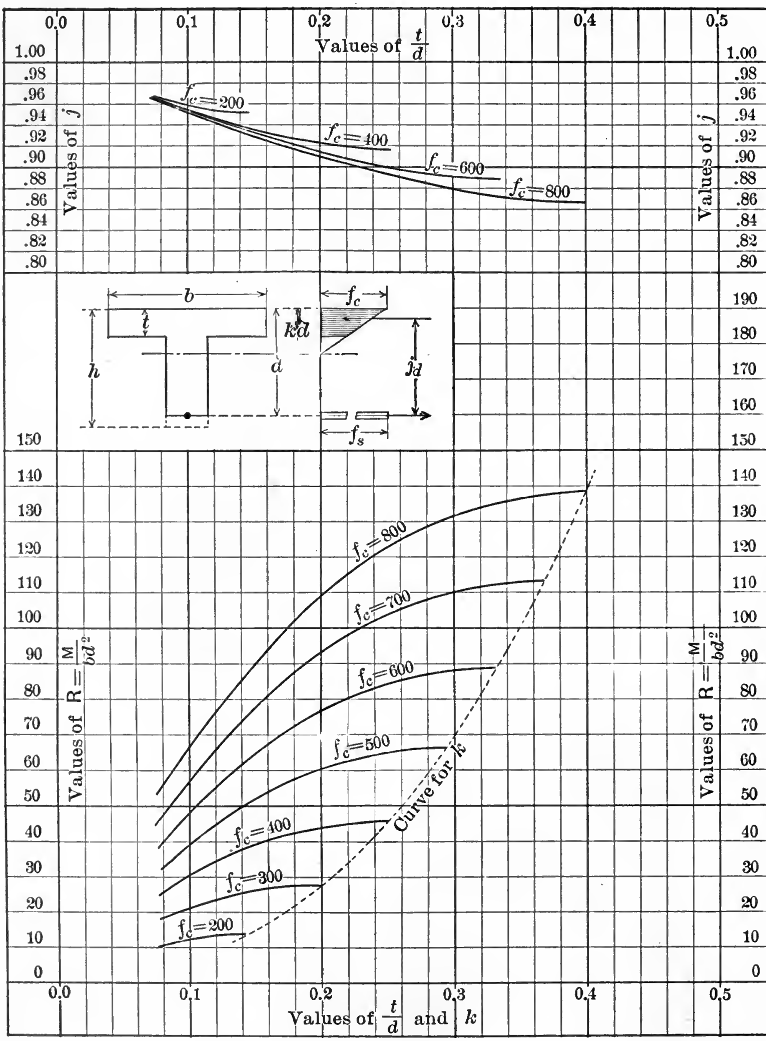


Plate VIII.—Coefficients of Resistance of T-beams.

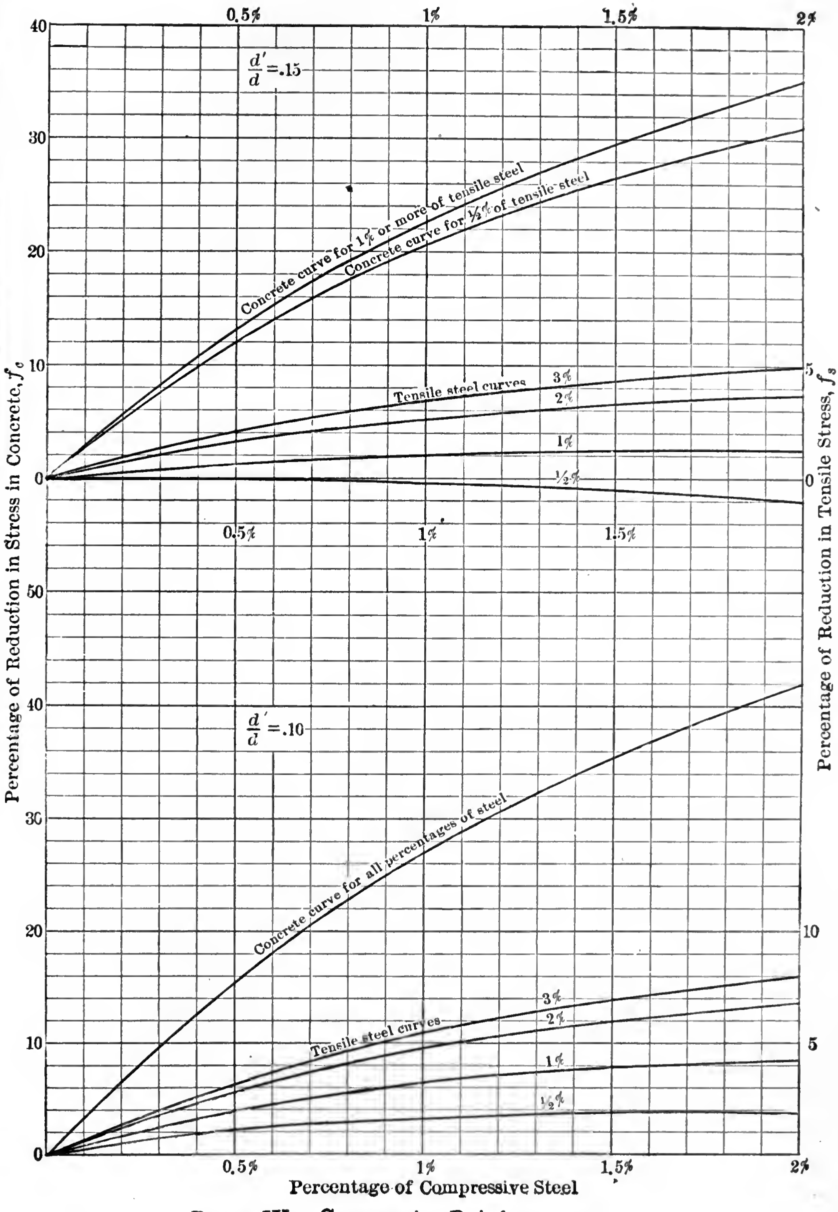


PLATE IX.—Compressive Reinforcement.

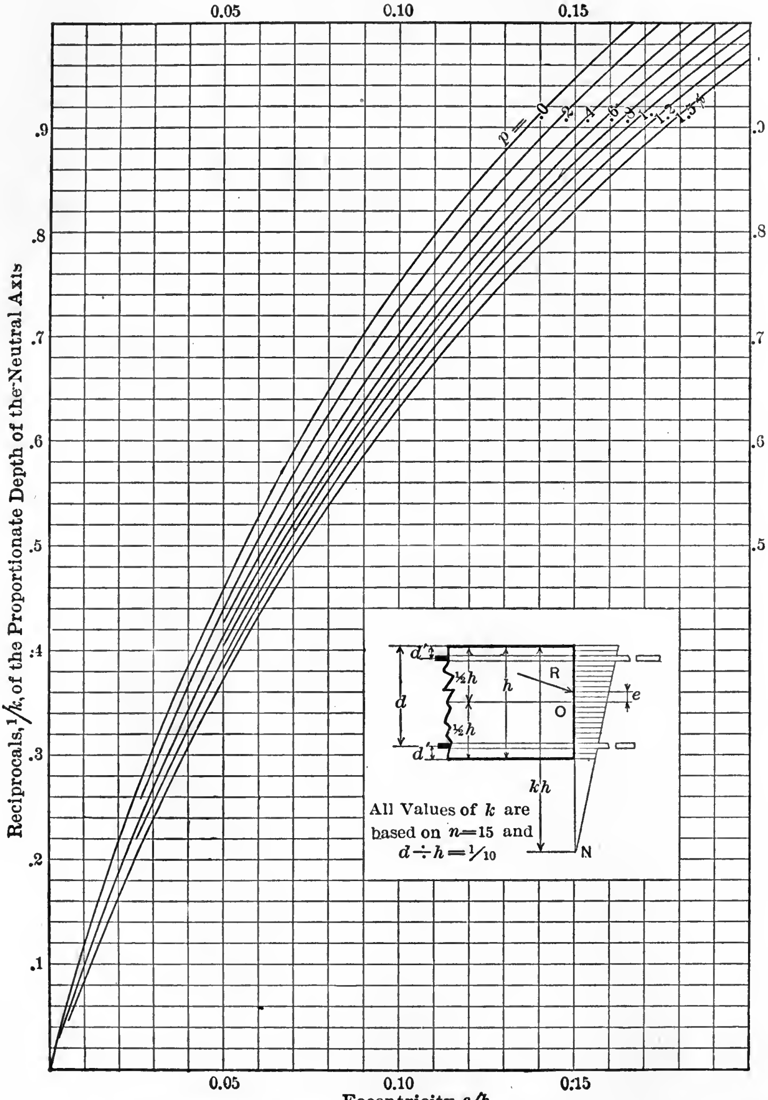


PLATE X.—Flexure and Direct Stress.

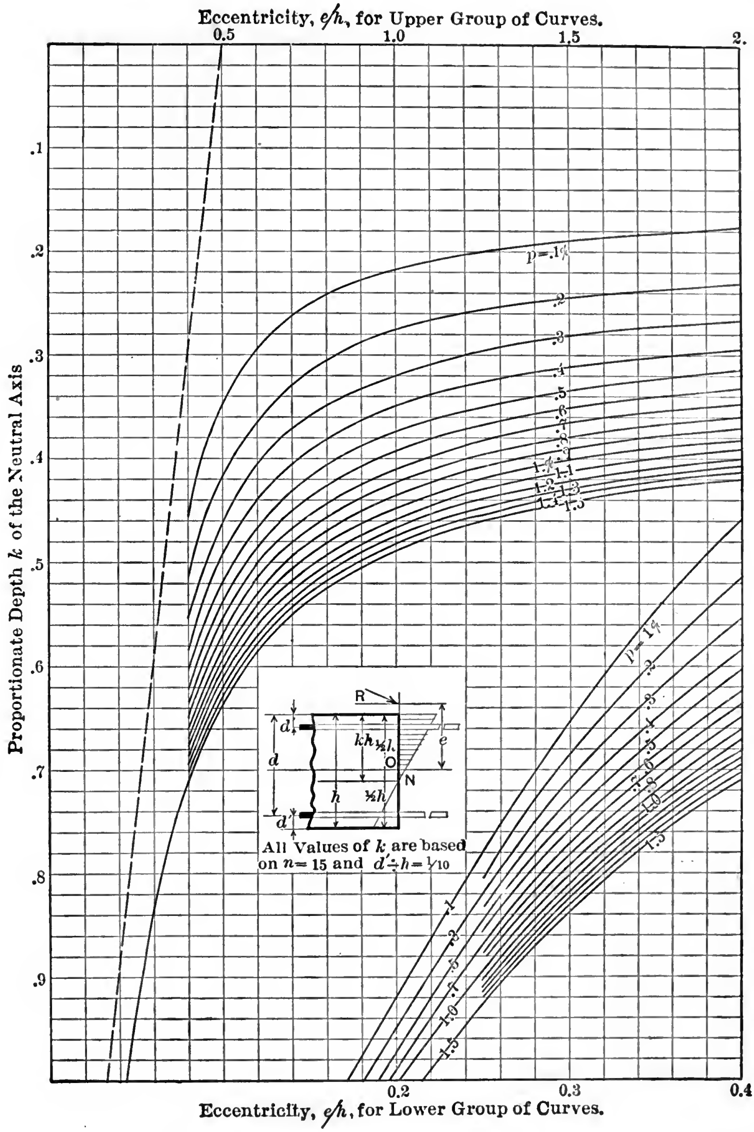


PLATE XI.—Flexure and Direct Stress.

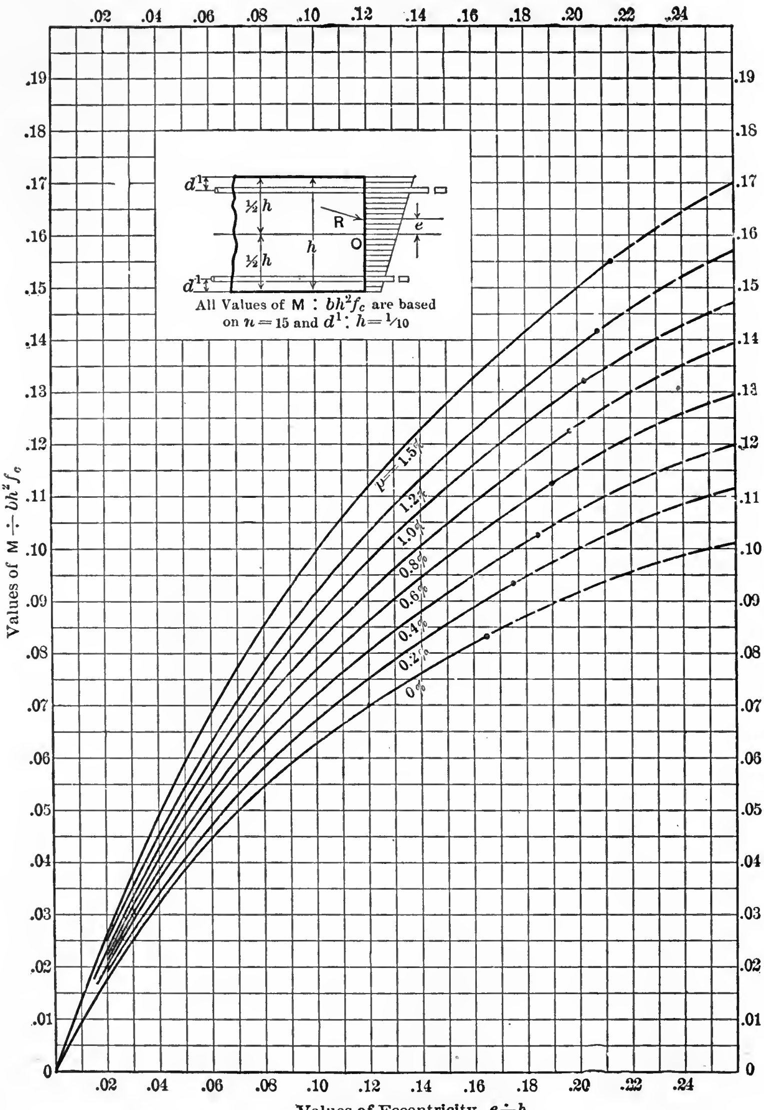


PLATE XII.—Flexure and Direct Stress.

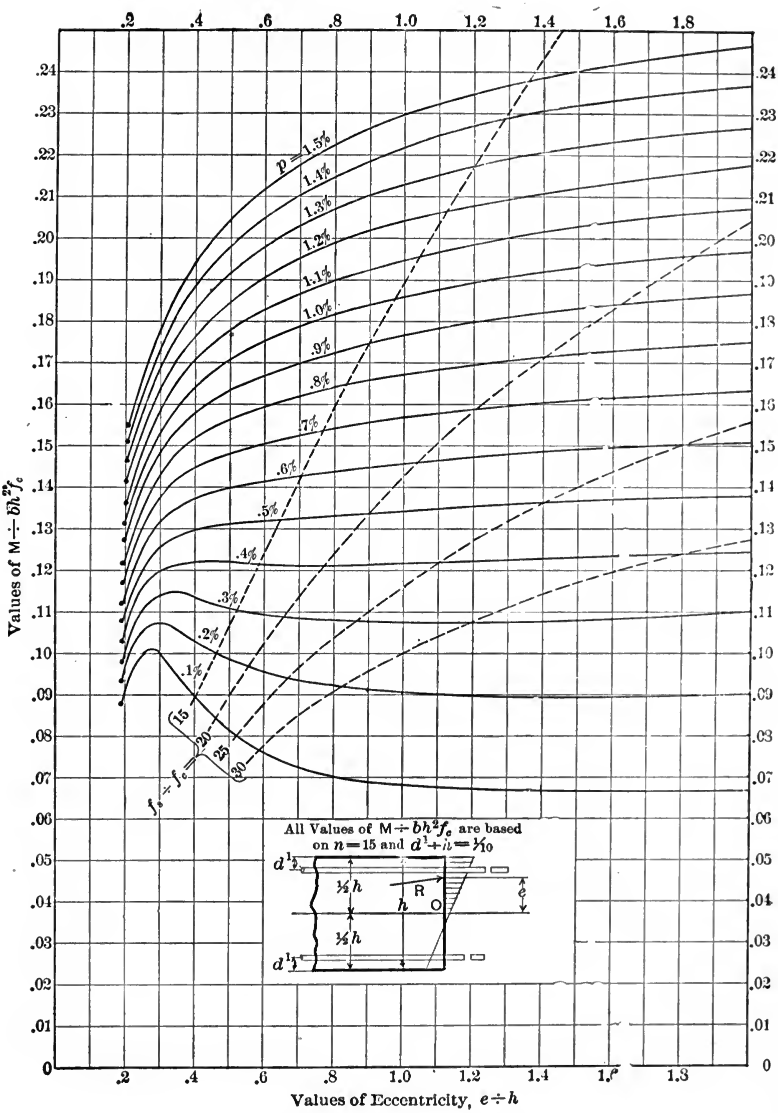


Plate XIII.—Flexure and Direct Stress.

Table No. 32.

AREAS, WEIGHTS, AND SPACING OF RODS.

ROUND RODS.

Diameter, Inches.	Area, Square Inches.	Circumference, Inches	Weight per Foot, Pounds.	Sectional Area of Steel per Foot of Slab when Spaced as follows:													
				2″	2½″	3″	3½″	4″	4½″	5″	5½″	6″	7″	8″	9″	10″	12″
¼	.0491	.7854	.167	.29	.23	.20	.17	.15	.13	.12	.11	.10	.08	.07	.07	.06	.05
5/16	.0767	.9818	.261	.46	.36	.31	.26	.23	.20	.18	.17	.15	.13	.11	.10	.09	.08
⅜	.1104	1.1781	.376	.66	.53	.44	.38	.33	.29	.26	.24	.22	.19	.17	.15	.13	.11
7/16	.1503	1.3745	.511	.90	.72	.60	.51	.45	.40	.36	.33	.30	.26	.23	.20	.18	.15
½	.1963	1.5708	.668	1.18	.94	.78	.67	.59	.52	.47	.43	.39	.34	.29	.26	.24	.20
9/16	.2485	1.7672	.845	1.49	1.19	.99	.85	.75	.66	.60	.54	.50	.43	.37	.33	.30	.25
⅝	.3068	1.9635	1.043	1.84	1.47	1.23	1.05	.92	.82	.74	.67	.61	.53	.46	.41	.37	.31
11/16	.3712	2.1599	1.262	2.23	1.78	1.48	1.27	1.11	.99	.89	.81	.74	.64	.56	.49	.45	.37
¾	.4418	2.3562	1.502	2.65	2.12	1.77	1.51	1.32	1.18	1.06	.96	.88	.76	.66	.59	.53	.44
13/16	.5185	2.5526	1.763	3.11	2.48	2.07	1.78	1.56	1.38	1.24	1.13	1.04	.89	.78	.69	.62	.52
⅞	.6013	2.7489	2.044	3.61	2.88	2.40	2.06	1.80	1.60	1.44	1.31	1.20	1.03	.90	.80	.72	.60
15/16	.6903	2.9453	2.347	4.14	3.31	2.76	2.37	2.07	1.84	1.66	1.51	1.38	1.18	1.03	.92	.83	.69
1	.7854	3.1416	2.670	4.71	3.77	3.14	2.69	2.36	2.09	1.88	1.71	1.57	1.35	1.18	1.05	.94	.78
1⅛	.9940	3.5343	3.380	5.96	4.77	3.98	3.41	2.98	2.65	2.39	2.17	1.99	1.70	1.49	1.33	1.19	.99
1¼	1.2272	3.9270	4.172	7.36	5.89	4.91	4.2'	3.68	3.27	2.95	2.68	2.45	2.10	1.84	1.64	1.47	1.23
1⅜	1.4849	4.3197	5.049	8.91	7.12	5.94	5.09	4.45	3.96	3.56	3.24	2.97	2.55	2.23	1.98	1.78	1.48
1½	1.7671	4.7124	6.008	10.60	8.48	7.07	6.06	5.30	4.71	4.24	3.86	3.53	3.03	2.65	2.36	2.12	1.77

TABLE NO. 35—*Continued.*

AREAS, WEIGHTS, AND SPACING OF RODS.

SQUARE RODS.

Dimension, Inches.	Area, Square Inches.	Perimeter, Inches.	Weight per Foot, Pounds.	Sectional Area of Steel per Foot of Slab when Spaced as follows:													
				2″	2½″	3″	3½″	4″	4½″	5″	5½″	6″	7″	8″	9″	10″	12″
¼	.0625	1.00	.212	.37	.30	.25	.21	.19	.17	.15	.13	.12	.11	.10	.08	.07	.06
5/16	.0977	1.25	.332	.59	.47	.39	.33	.29	.26	.23	.21	.19	.17	.15	.13	.12	.10
⅜	.1406	1.50	.478	.84	.67	.56	.48	.42	.37	.34	.31	.28	.24	.21	.19	.17	.14
7/16	.1914	1.75	.651	1.15	.92	.77	.66	.57	.51	.46	.42	.38	.33	.29	.25	.23	.19
½	.2500	2.00	.850	1.50	1.20	1.00	.86	.75	.67	.60	.55	.50	.43	.37	.33	.30	.25
9/16	.3164	2.25	1.076	1.90	1.52	1.27	1.08	.95	.84	.76	.69	.63	.54	.47	.42	.38	.32
⅝	.3906	2.50	1.328	2.34	1.87	1.56	1.34	1.17	1.04	.94	.85	.78	.67	.59	.52	.47	.39
11/16	.4727	2.75	1.607	2.84	2.27	1.99	1.62	1.42	1.33	1.13	1.03	.94	.81	.71	.66	.57	.47
¾	.5625	3.00	1.913	3.37	2.70	2.25	1.93	1.69	1.50	1.35	1.23	1.12	.96	.84	.75	.67	.56
13/16	.6602	3.25	2.245	3.96	3.17	2.64	2.26	1.98	1.76	1.58	1.44	1.32	1.13	.99	.88	.79	.66
⅞	.7656	3.50	2.603	4.59	3.67	3.06	2.62	2.30	2.04	1.84	1.67	1.53	1.31	1.15	1.02	.92	.77
15/16	.8789	3.75	2.988	5.27	4.22	3.52	3.01	2.64	2.34	2.11	1.92	1.76	1.51	1.32	1.17	1.05	.88
1	1.0000	4.00	3.400	6.00	4.80	4.00	3.43	3.00	2.67	2.40	2.18	2.00	1.71	1.50	1.33	1.20	1.00
1⅛	1.2656	4.50	4.303	7.59	6.08	5.06	4.34	3.80	3.37	3.04	2.76	2.53	2.17	1.89	1.69	1.52	1.27
1¼	1.5625	5.00	5.313	9.37	7.50	6.25	5.36	4.69	4.17	3.75	3.41	3.12	2.68	2.34	2.08	1.87	1.56
1⅜	1.8906	5.50	6.428	11.34	9.08	7.56	6.48	5.67	5.04	4.54	4.12	3.78	3.24	2.84	2.52	2.27	1.89
1½	2.2500	6.00	7.650	13.50	10.80	9.00	7.71	6.75	6.00	5.40	4.91	4.50	3.86	3.37	3.00	2.70	2.25

TABLE No. 36.

MATERIALS REQUIRED FOR ONE CUBIC YARD OF CONCRETE.

Proportion of Mixture.				Required for One Cubic Yard.		
Cement.	Sand.	Stone.	Ratio: Mortar/Stone	Cement, Barrels.	Sand, Cubic Yards.	Stone, Cubic Yards.
1	1	2.0	.72	2.75	0.39	0.78
1	1	2.5	.58	2.48	0.35	0.87
1	1	3.0	.48	2.25	0.31	0.94
1	1.5	2.5	.73	2.20	0.47	0.78
1	1.5	3.0	.61	2.00	0.42	0.84
1	1.5	3.5	.53	1.85	0.39	0.91
1	1.5	4.0	.46	1.72	0.36	0.96
1	2.0	3.0	.74	1.85	0.52	0.77
1	2.0	3.5	.63	1.72	0.48	0.83
1	2.0	4.0	.55	1.60	0.44	0.89
1	2.0	4.5	.49	1.48	0.42	0.93
1	2.5	4.0	.66	1.48	0.52	0.82
1	2.5	4.5	.58	1.38	0.48	0.87
1	2.5	5.0	.52	1.30	0.46	0.91
1	2.5	5.5	.47	1.22	0.43	0.94
1	3	4.5	.67	1.30	0.54	0.81
1	3	5.0	.60	1.22	0.51	0.85
1	3	5.5	.55	1.16	0.48	0.89
1	3	6.0	.50	1.10	0.46	0.92
1	3	6.5	.46	1.04	0.44	0.95

TABLE No. 37.—STRENGTH OF FLOOR-SLABS.

Bold-faced type, $M = \frac{1}{8}wl^2$; light-faced type, $M = \frac{1}{12}wl^2$.

1. $f_c = 500$ $f_s = 14{,}000$ $R = 77$ $p = .0062$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, lbs.	Span in Feet for Given Net Loads per Square Foot of Floor in Pounds.								
					50	75	100	150	200	250	300	400	500
2	¾	.094	1400	24.2	**3.6**	**3.1**	**2.8**	**2.4**	**2.1**	**1.9**	**1.7**	**1.5**	**1.4**
					4.4	3.8	3.4	2.9	2.6	2.3	2.1	1.8	1.7
2½	¾	.131	2800	30.3	**4.8**	**4.2**	**3.8**	**3.2**	**2.9**	**2.6**	**2.4**	**2.1**	**1.9**
					5.9	5.1	4.6	3.9	3.6	3.2	2.9	2.6	2.3
3	¾	.168	4700	36.4	**6.0**	**5.3**	**4.8**	**4.1**	**3.6**	**3.3**	**3.0**	**2.7**	**2.4**
					7.4	6.5	5.9	5.0	4.4	4.0	3.7	3.3	2.9
3½	¾	.206	7000	42.5	**7.1**	**6.3**	**5.7**	**4.9**	**4.4**	**4.0**	**3.7**	**3.3**	**2.9**
					8.7	7.7	7.0	6.0	5.4	4.9	4.5	4.0	3.6
4	1	.224	8300	48.5	**7.5**	**6.7**	**6.1**	**5.3**	**4.7**	**4.3**	**4.0**	**3.5**	**3.2**
					9.2	8.2	7.5	6.5	5.8	5.3	4.9	4.3	3.9
5	1	.299	14800	60.7	**9.5**	**8.5**	**7.9**	**6.8**	**6.2**	**5.7**	**5.2**	**4.6**	**4.2**
					11.6	10.4	9.7	8.3	7.6	7.0	6.4	5.6	5.1
6	1¼	.355	20900	72.9	**10.5**	**9.5**	**8.8**	**7.8**	**7.0**	**6.5**	**6.0**	**5.4**	**4.8**
					12.8	11.6	10.8	9.6	8.7	8.0	7.5	6.6	5.9
7	1¼	.430	30600	85.1	**12.2**	**11.1**	**10.3**	**9.3**	**8.3**	**7.7**	**7.2**	**6.5**	**5.8**
					14.9	13.6	12.6	11.4	10.2	9.4	8.8	8.0	7.1
8	1¼	.505	42100	97.4	**13.7**	**12.6**	**11.8**	**10.6**	**9.6**	**9.0**	**8.3**	**7.5**	**6.8**
					16.8	15.4	14.4	12.9	11.7	11.0	10.2	9.2	8.3
9	1½	.561	52000	109.5	**14.8**	**13.7**	**12.9**	**11.5**	**10.6**	**9.8**	**9.2**	**8.3**	**7.6**
					18.1	16.8	15.8	14.1	12.9	12.0	11.2	10.2	9.3
10	1½	.636	66800	121.7	**16.1**	**15.0**	**14.2**	**12.8**	**11.8**	**11.0**	**10.3**	**9.3**	**8.5**
					19.7	18.4	17.4	15.6	14.4	13.5	12.6	11.4	10.4
12	1½	.785	102000	145.9	**18.9**	**17.5**	**16.6**	**15.1**	**14.1**	**13.1**	**12.4**	**11.2**	**10.3**
					23.1	21.4	20.3	18.5	17.2	16.0	15.2	13.7	12.6

2. $f_c = 500$ $f_s = 15{,}000$ $R = 74$ $p = .0056$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, lbs.	50	75	100	150	200	250	300	400	500
2	¾	.083	1400	24.2	**3.5**	**3.1**	**2.7**	**2.3**	**2.0**	**1.8**	**1.7**	**1.5**	**1.3**
					4.3	3.8	3.3	2.8	2.4	2.2	2.1	1.8	1.6
2½	¾	.117	2700	30.3	**4.7**	**4.1**	**3.7**	**3.2**	**2.8**	**2.5**	**2.3**	**2.0**	**1.8**
					5.8	5.0	4.5	3.9	3.4	3.1	2.8	2.4	2.2
3	¾	.150	4500	36.4	**5.9**	**5.2**	**4.7**	**4.0**	**3.6**	**3.2**	**3.0**	**2.6**	**2.4**
					7.2	6.4	5.8	4.9	4.4	3.9	3.7	3.2	2.9
3½	¾	.183	6700	42.5	**7.0**	**6.2**	**5.6**	**4.8**	**4.3**	**3.9**	**3.6**	**3.2**	**2.9**
					8.6	7.6	6.9	5.9	5.3	4.8	4.4	3.9	3.6
4	1	.200	8000	48.5	**7.4**	**6.6**	**6.0**	**5.2**	**4.6**	**4.2**	**3.9**	**3.4**	**3.1**
					9.0	8.1	7.4	6.4	5.6	5.1	4.8	4.2	3.8
5	1	.267	14200	60.6	**9.2**	**8.3**	**7.7**	**6.7**	**6.0**	**5.5**	**5.1**	**4.5**	**4.1**
					11.2	10.2	9.4	8.2	7.4	6.7	6.2	5.5	5.0
6	1¼	.317	20100	72.8	**10.4**	**9.5**	**8.8**	**7.7**	**7.0**	**6.4**	**6.0**	**5.3**	**4.8**
					12.7	11.6	10.8	9.4	8.6	7.8	7.4	6.5	5.9
7	1¼	.383	29400	84.9	**12.0**	**11.1**	**10.3**	**9.1**	**8.3**	**7.6**	**7.1**	**6.4**	**5.8**
					14.7	13.6	12.6	11.1	10.2	9.3	8.7	7.8	7.1
8	1¼	.450	40500	97.2	**13.5**	**12.5**	**11.7**	**10.5**	**9.5**	**8.8**	**8.2**	**7.4**	**6.7**
					16.5	15.3	14.3	12.8	11.6	10.8	10.0	9.0	8.2
9	1½	.500	50000	109.3	**14.5**	**13.5**	**12.6**	**11.3**	**10.4**	**9.6**	**9.0**	**8.1**	**7.4**
					17.7	16.5	15.4	13.8	12.7	11.7	11.0	9.9	9.0
10	1½	.567	64200	121.5	**15.8**	**14.7**	**13.9**	**12.6**	**11.5**	**10.7**	**10.1**	**9.1**	**8.3**
					19.3	18.0	17.0	15.4	14.1	13.1	12.4	11.1	10.2
12	1½	.700	98000	145.7	**18.2**	**17.2**	**16.3**	**14.8**	**13.7**	**12.8**	**12.1**	**10.9**	**10.0**
					22.2	21.0	19.9	18.1	16.7	15.7	14.8	13.3	12.2

TABLE NO. 37 (*Continued*).—STRENGTH OF FLOOR-SLABS.

Bold-faced type, $M=\frac{1}{8}wl^2$; light-faced type, $M=\frac{1}{12}wl^2$.

3. $f_c=500$ $f_s=16,000$ $R=71$ $p=.0050$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	Span in Feet for Given Net Loads per Square Foot of Floor in Pounds. 50	75	100	150	200	250	300	400	500
2	$\frac{3}{4}$	.075	1300	24.2	**3.5**	**3.0**	**2.7**	**2.3**	**2.0**	**1.8**	**1.7**	**1.4**	**1.3**
					4.3	3.7	3.3	2.8	2.4	2.2	2.1	1.7	1.6
2½	$\frac{3}{4}$	.105	2600	30.3	**4.7**	**4.1**	**3.7**	**3.1**	**2.8**	**2.5**	**2.3**	**2.0**	**1.8**
					5.8	5.0	4.5	3.8	3.4	3.1	2.8	2.4	2.2
3	$\frac{3}{4}$	.135	4300	36.4	**5.8**	**5.1**	**4.6**	**4.0**	**3.5**	**3.2**	**2.9**	**2.6**	**2.3**
					7.1	6.2	5.6	5.0	4.3	3.9	3.6	3.2	2.8
3½	$\frac{3}{4}$	.165	6500	42.5	**6.8**	**6.1**	**5.5**	**4.8**	**4.2**	**3.8**	**3.6**	**3.1**	**2.8**
					8.3	7.5	6.7	5.9	5.1	4.6	4.4	3.8	3.4
4	1	.180	7700	48.5	**7.2**	**6.4**	**5.9**	**5.1**	**4.5**	**4.1**	**3.8**	**3.4**	**3.1**
					8.8	7.8	7.2	6.2	5.5	5.1	4.6	4.2	3.8
5	1	.239	13700	60.7	**9.1**	**8.2**	**7.5**	**6.6**	**5.9**	**5.4**	**5.0**	**4.4**	**4.0**
					11.1	10.0	9.2	8.1	7.2	6.6	6.1	5.4	4.9
6	1¼	.284	19300	72.7	**10.3**	**9.4**	**8.7**	**7.6**	**6.9**	**6.3**	**5.9**	**5.2**	**4.7**
					12.6	11.5	10.6	9.3	8.4	7.7	7.2	6.4	5.8
7	1¼	.344	28300	84.8	**11.9**	**10.9**	**10.1**	**9.0**	**8.2**	**7.5**	**7.0**	**6.3**	**5.7**
					14.5	13.3	12.4	11.0	10.0	9.2	8.6	7.7	7.0
8	1¼	.404	39000	97.0	**13.4**	**12.3**	**11.5**	**10.3**	**9.4**	**8.7**	**8.1**	**7.3**	**6.6**
					16.4	15.0	14.1	12.6	11.5	10.6	9.9	8.9	8.1
9	1½	.449	48100	109.1	**14.3**	**13.3**	**12.4**	**11.2**	**10.2**	**9.5**	**8.9**	**8.0**	**7.3**
					17.5	16.3	15.2	13.7	12.5	11.6	10.9	9.8	8.9
10	1½	.509	61800	121.3	**15.6**	**13.6**	**13.7**	**12.4**	**11.4**	**10.6**	**9.9**	**8.9**	**8.2**
					19.1	17.9	16.7	15.2	13.9	12.9	12.1	10.9	10.0
12	1½	.629	94400	145.5	**17.9**	**16.9**	**16.0**	**14.6**	**13.5**	**12.6**	**11.9**	**10.7**	**9.9**
					21.8	20.7	19.5	17.9	16.5	15.4	14.5	13.1	12.1

4. $f_c=500$ $f_s=18,000$ $R=66$ $p=.0041$

2	$\frac{3}{4}$	.061	1200	24.2	**3.3**	**2.9**	**2.6**	**2.2**	**1.9**	**1.7**	**1.6**	**1.4**	**1.3**
					4.0	3.6	3.2	2.7	2.3	2.1	2.0	1.7	1.6
2½	$\frac{3}{4}$	.086	2400	30.3	**4.5**	**3.9**	**3.5**	**3.0**	**2.6**	**2.4**	**2.2**	**1.9**	**1.7**
					5.5	4.8	4.3	3.7	3.2	2.9	2.7	2.3	2.1
3	$\frac{3}{4}$	.110	4000	36.4	**5.6**	**4.9**	**4.4**	**3.8**	**3.4**	**3.0**	**2.8**	**2.5**	**2.2**
					6.9	6.0	5.4	4.6	4.2	3.7	3.4	3.1	2.7
3½	$\frac{3}{4}$	.135	6000	42.4	**6.6**	**5.9**	**5.3**	**4.6**	**4.0**	**3.7**	**3.4**	**3.0**	**2.7**
					8.1	7.2	6.5	5.6	4.9	4.5	4.2	3.7	3.3
4	1	.147	7200	48.4	**7.0**	**6.2**	**5.7**	**4.9**	**4.4**	**4.0**	**3.7**	**3.3**	**2.9**
					8.6	7.6	7.0	6.0	5.4	4.9	4.5	4.0	3.6
5	1	.196	12700	60.5	**8.8**	**7.9**	**7.3**	**6.4**	**5.7**	**5.2**	**4.8**	**4.3**	**3.9**
					10.8	9.7	8.9	7.8	7.0	6.4	5.9	5.3	4.8
6	1¼	.233	18000	72.6	**9.8**	**9.0**	**8.3**	**7.3**	**6.6**	**6.1**	**5.7**	**5.0**	**4.6**
					12.0	11.0	10.2	8.9	8.1	7.5	7.0	6.1	5.6
7	1¼	.282	26300	84.7	**11.4**	**10.4**	**9.7**	**8.6**	**7.8**	**7.2**	**6.7**	**6.0**	**5.5**
					13.9	12.7	11.9	10.5	9.6	8.8	8.2	7.4	6.7
8	1¼	.331	36300	96.8	**12.8**	**11.8**	**11.0**	**9.9**	**9.0**	**8.3**	**7.8**	**7.0**	**6.4**
					15.6	14.4	13.5	12.1	11.0	10.2	9.6	8.6	7.8
9	1½	.368	44800	108.9	**13.6**	**12.7**	**11.9**	**10.7**	**9.8**	**9.1**	**8.5**	**7.7**	**7.0**
					16.6	15.5	14.5	13.1	12.0	11.1	10.4	9.4	8.6
10	1½	.417	57500	121.1	**14.9**	**13.9**	**13.1**	**11.9**	**10.9**	**10.1**	**9.5**	**8.6**	**7.9**
					18.2	17.0	16.0	14.5	13.3	12.4	11.6	10.5	9.7
12	1½	.515	87700	145.3	**17.1**	**16.2**	**15.4**	**14.0**	**13.0**	**12.1**	**11.4**	**10.4**	**9.5**
					20.9	19.8	18.8	17.1	15.9	14.8	13.9	12.7	11.6

TABLE No. 37 (*Continued*).—STRENGTH OF FLOOR-SLABS.

Bold-faced type, $M=\frac{1}{8}wl^2$; light-faced type, $M=\frac{1}{12}wl^2$.

5. $f_c=600$ $f_s=14{,}000$ $R=102$ $p=.0084$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	Span in Feet for Given Net Loads per Square Foot of Floor in Pounds.								
					50	75	100	150	200	250	300	400	500
2	¾	.126	1900	24.3	**4.1**	**3.6**	**3.2**	**2.7**	**2.4**	**2.2**	**2.0**	**1.7**	**1.6**
					5.0	4.4	3.9	3.3	2.9	2.7	2.4	2.1	2.0
2½	¾	.176	3800	30.4	**5.6**	**4.9**	**4.4**	**3.7**	**3.3**	**3.0**	**2.7**	**2.4**	**2.2**
					6.9	6.0	5.4	4.5	4.0	3.7	3.3	2.9	2.7
3	¾	.226	6200	36.5	**6.9**	**6.1**	**5.5**	**4.7**	**4.2**	**3.8**	**3.5**	**3.1**	**2.8**
					8.4	7.5	6.7	5.8	5.1	4.6	4.3	3.8	3.4
3½	¾	.277	9300	42.7	**8.2**	**7.2**	**6.6**	**5.7**	**5.0**	**4.6**	**4.2**	**3.7**	**3.4**
					10.0	8.8	8.1	7.0	6.1	5.6	5.1	4.5	4.2
4	1	.302	11000	48.7	**8.6**	**7.7**	**7.0**	**6.1**	**5.4**	**5.0**	**4.6**	**4.0**	**3.7**
					10.5	9.4	8.6	7.5	6.6	6.1	5.6	4.9	4.5
5	1	.403	19600	60.9	**10.8**	**9.8**	**9.0**	**7.9**	**7.1**	**6.5**	**6.0**	**5.3**	**4.8**
					13.2	12.0	11.0	9.7	8.7	8.0	7.4	6.5	5.9
6	1¼	.478	27700	73.1	**12.2**	**11.2**	**10.3**	**9.1**	**8.2**	**7.5**	**7.0**	**6.2**	**5.7**
					14.9	13.7	12.6	11.1	10.0	9.2	8.6	7.6	7.0
7	1¼	.579	40600	85.4	**14.1**	**13.0**	**12.1**	**10.7**	**9.7**	**9.0**	**8.4**	**7.5**	**6.8**
					17.2	15.9	14.8	13.1	11.9	11.0	10.3	9.2	8.3
8	1¼	.680	55900	97.8	**15.9**	**14.7**	**13.7**	**12.2**	**11.2**	**10.3**	**9.7**	**8.6**	**7.9**
					19.4	18.0	16.8	14.9	13.7	12.6	11.9	10.5	9.7
9	1½	.755	69000	109.8	**16.9**	**15.7**	**14.8**	**13.3**	**12.2**	**11.3**	**10.6**	**9.5**	**8.7**
					20.6	19.2	18.1	16.2	14.9	13.8	12.9	11.6	10.6
10	1½	.856	88600	122.0	**18.5**	**17.3**	**16.3**	**14.7**	**13.5**	**12.6**	**11.8**	**10.6**	**9.7**
					22.6	21.1	19.9	18.0	16.5	15.4	14.4	12.9	11.9
12	1½	1.057	135300	146.4	**21.4**	**20.1**	**19.1**	**17.4**	**16.1**	**15.0**	**14.2**	**12.8**	**11.8**
					26.2	24.6	23.3	21.3	19.7	18.3	17.3	15.7	14.4

6. $f_c=600$ $f_s=15{,}000$ $R=98$ $p=.0075$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	50	75	100	150	200	250	300	400	500
2	¾	.112	1800	24.2	**4.1**	**3.5**	**3.2**	**2.7**	**2.3**	**2.1**	**1.9**	**1.7**	**1.5**
					5.0	4.3	3.9	3.3	2.8	2.6	2.3	2.1	1.8
2½	¾	.157	3600	30.4	**5.5**	**4.8**	**4.3**	**3.7**	**3.2**	**2.9**	**2.7**	**2.4**	**2.1**
					6.7	5.9	5.3	4.5	3.9	3.6	3.3	2.9	2.6
3	¾	.202	6000	36.5	**6.8**	**6.0**	**5.4**	**4.6**	**4.1**	**3.7**	**3.4**	**3.0**	**2.7**
					8.3	7.4	6.6	5.6	5.0	4.5	4.2	3.7	3.3
3½	¾	.247	8900	42.7	**8.0**	**7.1**	**6.5**	**5.6**	**5.0**	**4.5**	**4.2**	**3.7**	**3.3**
					9.8	8.7	8.0	6.9	6.1	5.5	5.1	4.5	4.0
4	1	.270	10600	48.7	**8.5**	**7.6**	**6.9**	**6.0**	**5.3**	**4.9**	**4.5**	**4.0**	**3.6**
					10.4	9.3	8.4	7.4	6.5	6.0	5.5	4.9	4.4
5	1	.360	18900	60.9	**10.7**	**9.6**	**8.8**	**7.7**	**7.0**	**6.4**	**5.9**	**5.2**	**4.7**
					13.1	11.7	10.8	9.4	8.6	7.8	7.2	6.4	5.8
6	1¼	.427	26700	73.1	**12.0**	**11.0**	**10.1**	**8.9**	**8.1**	**7.4**	**6.9**	**6.1**	**5.6**
					14.7	13.5	12.4	10.9	9.9	9.0	8.4	7.5	6.9
7	1¼	.517	39100	85.5	**13.9**	**12.8**	**11.9**	**10.5**	**9.6**	**8.8**	**8.2**	**7.3**	**6.7**
					17.0	15.7	14.5	12.8	11.7	10.8	10.0	8.9	8.2
8	1¼	.607	53800	97.9	**15.6**	**14.4**	**13.5**	**12.1**	**11.0**	**10.2**	**9.5**	**8.5**	**7.8**
					19.1	17.6	16.5	14.8	13.5	12.5	11.6	10.4	9.6
9	1½	.675	66400	109.6	**16.7**	**15.5**	**14.6**	**13.1**	**12.0**	**11.1**	**10.4**	**9.4**	**8.5**
					20.4	19.0	17.8	16.0	14.7	13.6	12.7	11.5	10.4
10	1½	.765	85300	121.8	**18.2**	**17.0**	**16.0**	**14.5**	**13.3**	**12.4**	**11.6**	**10.4**	**9.6**
					22.2	20.8	19.5	17.7	16.2	15.2	14.2	12.7	11.7
12	1½	.945	130200	146.2	**21.1**	**19.8**	**18.8**	**17.1**	**15.8**	**14.8**	**14.0**	**12.6**	**11.6**
					25.8	24.2	23.0	20.9	19.3	18.1	17.1	15.4	14.2

TABLE No. 37 (*Continued*).—STRENGTH OF FLOOR-SLABS.

Bold-faced type, $M=\frac{1}{8}wl^2$; light-faced type, $M=\frac{1}{12}wl^2$.

7. $f_c=600$ $f_s=16,000$ $R=95$ $p=.0068$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	Span in Feet for Given Net Loads per Square Foot of Floor in Pounds.								
					50	75	100	150	200	250	300	400	500
2	¾	.101	1800	24.2	**4.0**	**3.5**	**3.1**	**2.6**	**2.3**	**2.1**	**1.9**	**1.7**	**1.5**
					4.9	4.3	3.8	3.2	2.8	2.6	2.3	2.1	1.8
2½	¾	.142	3500	30.3	**5.4**	**4.7**	**4.2**	**3.6**	**3.2**	**2.9**	**2.7**	**2.3**	**2.1**
					6.6	5.8	5.1	4.4	3.9	3.6	3.3	2.8	2.6
3	¾	.182	5800	36.4	**6.7**	**5.9**	**5.3**	**4.5**	**4.0**	**3.7**	**3.4**	**3.0**	**2.7**
					8.2	7.2	6.5	5.5	4.9	4.5	4.2	3.7	3.3
3½	¾	.223	8600	42.6	**7.9**	**7.0**	**6.3**	**5.5**	**4.9**	**4.4**	**4.1**	**3.6**	**3.3**
					9.7	8.6	7.7	6.7	6.0	5.4	5.0	4.4	4.0
4	1	.243	10300	48.6	**8.3**	**7.4**	**6.8**	**5.9**	**5.2**	**4.8**	**4.4**	**3.9**	**3.5**
					10.2	9.0	8.3	7.2	6.4	5.9	5.4	4.8	4.3
5	1	.324	18300	60.8	**10.5**	**9.5**	**8.7**	**7.6**	**6.8**	**6.3**	**5.8**	**5.1**	**4.7**
					12.8	11.6	10.6	9.3	8.3	7.7	7.1	6.2	5.8
6	1¼	.385	25700	73.0	**11.8**	**10.7**	**9.9**	**8.7**	**7.9**	**7.3**	**6.8**	**6.0**	**5.5**
					14.4	13.1	12.1	10.6	9.7	8.9	8.3	7.4	6.7
7	1¼	.466	37700	85.2	**13.6**	**12.5**	**11.6**	**10.3**	**9.4**	**8.7**	**8.1**	**7.2**	**6.5**
					16.6	15.3	14.2	12.6	11.5	10.6	9.9	8.8	8.0
8	1¼	.547	52000	97.7	**15.3**	**14.2**	**13.2**	**11.8**	**10.8**	**10.0**	**9.4**	**8.3**	**7.6**
					18.7	17.3	16.1	14.4	13.2	12.2	11.5	10.2	9.3
9	1½	.608	64200	109.4	**16.4**	**15.2**	**14.3**	**12.8**	**11.8**	**10.9**	**10.2**	**9.2**	**8.4**
					20.0	18.6	17.5	15.7	14.4	13.3	12.5	11.2	10.3
10	1½	.689	82400	121.6	**17.9**	**16.7**	**15.7**	**14.2**	**13.1**	**12.2**	**11.4**	**10.2**	**9.4**
					21.9	20.4	19.2	17.3	16.0	14.9	13.9	12.5	11.5
12	1½	.851	125800	146.2	**20.6**	**19.4**	**18.4**	**16.8**	**15.5**	**14.5**	**13.7**	**12.4**	**11.4**
					25.2	23.7	22.5	20.5	19.0	17.7	16.8	15.2	13.9

8. $f_c=600$ $f_s=18,000$ $R=89$ $p=.0056$

2	¾	.083	1700	24.1	**3.9**	**3.3**	**3.0**	**2.5**	**2.2**	**2.0**	**1.8**	**1.6**	**1.4**
					4.8	4.0	3.7	3.1	2.7	2.4	2.2	2.0	1.7
2½	¾	.117	3300	30.3	**5.2**	**4.5**	**4.1**	**3.5**	**3.1**	**2.8**	**2.6**	**2.2**	**2.0**
					6.4	5.5	5.0	4.3	3.8	3.4	3.2	2.7	2.4
3	¾	.150	5400	36.4	**6.5**	**5.7**	**5.1**	**4.4**	**3.9**	**3.5**	**3.3**	**2.9**	**2.6**
					8.0	7.0	6.2	5.4	4.8	4.3	4.0	3.6	3.2
3½	¾	.183	8100	42.6	**7.6**	**6.8**	**6.1**	**5.3**	**4.7**	**4.3**	**4.0**	**3.5**	**3.1**
					9.3	8.3	7.5	6.5	5.8	5.3	4.9	4.3	3.8
4	1	.200	9600	48.6	**8.1**	**7.2**	**6.6**	**5.7**	**5.1**	**4.6**	**4.3**	**3.8**	**3.4**
					9.9	8.8	8.1	7.0	6.2	5.6	5.3	4.6	4.2
5	1	.267	17100	60.7	**10.1**	**9.2**	**8.4**	**7.4**	**6.6**	**6.1**	**5.6**	**5.0**	**4.5**
					12.4	11.2	10.3	9.0	8.1	7.5	6.9	6.1	5.5
6	1¼	.317	24100	72.8	**11.4**	**10.4**	**9.6**	**8.5**	**7.7**	**7.1**	**6.6**	**5.8**	**5.3**
					13.9	12.7	11.7	10.4	9.4	8.7	8.1	7.1	6.5
7	1¼	.383	35300	85.2	**13.2**	**12.1**	**11.3**	**10.0**	**9.1**	**8.4**	**7.8**	**7.0**	**6.3**
					16.1	14.8	13.8	12 2	11.1	10.3	9.6	8.6	7.7
8	1¼	.450	48600	97.5	**14.8**	**13.7**	**12.8**	**11.4**	**10.4**	**9.6**	**9.0**	**8.1**	**7.4**
					18.1	16.8	15.7	13.9	12.7	11.7	11.0	9.9	9.0
9	1½	.500	60000	109.2	**15.9**	**14.7**	**13.8**	**12.4**	**11.4**	**10.6**	**9.9**	**8.9**	**8.1**
					19.4	18.0	16.9	15.2	13.9	12.9	12.1	10.9	9.9
10	1½	.567	77100	121.3	**17.3**	**16.2**	**15.2**	**13.8**	**12.6**	**11.8**	**11.0**	**9.9**	**9.1**
					21.1	19.8	18.6	16.9	15.4	14.4	13.5	12.1	11.1
12	1½	.700	117600	145.7	**20.0**	**18.8**	**17.8**	**16.3**	**15.1**	**14.1**	**13.3**	**12.0**	**11.0**
					24.5	23.0	21.7	19.9	18.5	17.2	16.2	14.7	13.5

TABLE No. 37 (*Continued*).—STRENGTH OF FLOOR-SLABS.

Bold-faced type, $M=\frac{1}{8}wl^2$; light-faced type, $M=\frac{1}{12}wl^2$.

9. $f_c=700$ $f_s=14,000$ $R=129$ $p=.0107$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	Span in Feet for Given Net Loads per Square Foot of Floor in Pounds. 50	75	100	150	200	250	300	400	500
2	¾	.161	2400	24.4	**4.7**	**4.0**	**3.6**	**3.0**	**2.7**	**2.4**	**2.2**	**1.9**	**1.7**
					5.8	4.9	4.4	3.7	3.3	2.9	2.7	2.3	2.1
2½	¾	.225	4700	30.6	**6.3**	**5.5**	**4.9**	**4.2**	**3.7**	**3.4**	**3.1**	**2.7**	**2.4**
					7.7	6.7	6.0	5.1	4.5	4.2	3.8	3.3	2.9
3	¾	.289	7800	36.7	**7.8**	**6.8**	**6.2**	**5.3**	**4.7**	**4.3**	**3.9**	**3.5**	**3.1**
					9.6	8.3	7.6	6.5	5.8	5.3	4.8	4.3	3.8
3½	¾	.354	11700	43.0	**9.2**	**8.1**	**7.4**	**6.4**	**5.7**	**5.2**	**4.8**	**4.2**	**3.8**
					11.2	9.9	9.0	7.8	7.0	6.4	5.9	5.1	4.6
4	1	.386	13900	48.9	**9.7**	**8.6**	**7.9**	**6.8**	**6.1**	**5.6**	**5.1**	**4.5**	**4.1**
					11.9	10.5	9.7	8.3	7.5	6.9	6.2	5.5	5.0
5	1	.514	24700	61.1	**12.2**	**11.0**	**10.1**	**8.8**	**7.9**	**7.3**	**6.7**	**6.0**	**5.4**
					14.9	13.5	12.4	10.8	9.7	8.9	8.2	7.4	6.6
6	1¼	.611	34800	73.5	**13.7**	**12.5**	**11.5**	**10.2**	**9.2**	**8.5**	**7.9**	**7.0**	**6.4**
					16.8	15.3	14.1	12.5	11.2	10.4	9.7	8.6	7.8
7	1¼	.739	51000	85.7	**15.8**	**14.6**	**13.5**	**12.0**	**10.9**	**10.1**	**9.4**	**8.4**	**7.6**
					19.3	17.8	16.5	14.7	13.3	12.4	11.5	10.3	9.3
8	1¼	.868	70300	98.1	**17.8**	**16.5**	**15.4**	**13.7**	**12.5**	**11.6**	**10.9**	**9.7**	**8.9**
					21.7	20.1	18.8	16.8	15.3	14.2	13.3	11.9	10.9
9	1½	.964	86800	110.3	**19.0**	**17.7**	**16.6**	**14.9**	**13.7**	**12.7**	**11.9**	**10.6**	**9.7**
					23.2	21.6	20.3	18.2	16.8	15.5	14.5	12.9	11.9
10	1½	1.093	111500	122.6	**20.8**	**19.4**	**18.3**	**16.5**	**15.2**	**14.1**	**13.3**	**11.9**	**10.9**
					25.5	23.7	22.3	20.1	18.6	17.2	16.2	14.5	13.3
12	1½	1.350	170100	147.3	**24.0**	**22.6**	**21.5**	**19.6**	**18.1**	**16.9**	**15.9**	**14.4**	**13.2**
					29.3	27.6	26.3	24.0	22.1	20.6	19.4	17.6	16.1

10. $f_c=700$ $f_s=15,000$ $R=124$ $p=.0096$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	50	75	100	150	200	250	300	400	500
2	¾	.144	2300	24.4	**4.6**	**4.0**	**3.5**	**3.0**	**2.6**	**2.4**	**2.2**	**1.9**	**1.7**
					5.6	4.9	4.3	3.7	3.2	2.9	2.7	2.3	2.1
2½	¾	.202	4600	30.6	**6.1**	**5.4**	**4.8**	**4.1**	**3.6**	**3.3**	**3.0**	**2.7**	**2.4**
					7.5	6.6	5.9	5.0	4.4	4.0	3.7	3.3	2.9
3	¾	.259	7600	36.7	**7.6**	**6.7**	**6.1**	**5.2**	**4.6**	**4.2**	**3.9**	**3.4**	**3.1**
					9.3	8.2	7.5	6.4	5.6	5.1	4.8	4.2	3.8
3½	¾	.317	11300	42.9	**9.0**	**8.0**	**7.3**	**6.3**	**5.6**	**5.1**	**4.7**	**4.1**	**3.7**
					11.0	9.8	8.9	7.7	6.9	6.2	5.8	5.0	4.5
4	1	.346	13400	48.8	**9.5**	**8.5**	**7.8**	**6.7**	**6.0**	**5.6**	**5.1**	**4.5**	**4.0**
					11.6	10.4	9.6	8.2	7.4	6.9	6.2	5.5	4.9
5	1	.461	23900	61.1	**12.0**	**10.8**	**9.9**	**8.7**	**7.8**	**7.1**	**6.6**	**5.9**	**5.3**
					14.7	13.2	12.1	10.6	9.6	8.7	8.1	7.2	6.5
6	1¼	.548	33700	73.3	**13.5**	**12.3**	**11.4**	**10.0**	**9.1**	**8.3**	**7.7**	**6.9**	**6.3**
					16.5	15.0	13.9	12.2	11.1	10.2	9.4	8.4	7.7
7	1¼	.663	49300	85.5	**15.6**	**14.3**	**13.4**	**11.8**	**10.7**	**9.9**	**9.2**	**8.2**	**7.5**
					19.1	17.5	16.4	14.4	13.1	12.1	11.2	10.0	9.2
8	1¼	.778	68000	97.9	**17.5**	**16.2**	**15.1**	**13.5**	**12.3**	**11.4**	**10.7**	**9.5**	**8.7**
					21.4	19.8	18.5	16.5	15.0	13.9	13.1	11.6	10.6
9	1½	.865	83900	110.1	**18.7**	**17.4**	**16.3**	**14.7**	**13.4**	**12.5**	**11.7**	**10.5**	**9.6**
					22.9	21.3	19.9	18.0	16.4	15.3	14.3	12.8	11.7
10	1½	.980	107800	122.3	**20.4**	**19.1**	**18.0**	**16.2**	**14.9**	**13.9**	**13.0**	**11.7**	**10.7**
					25.0	23.3	22.0	19.3	18.2	17.0	15.9	14.3	13.1
12	1½	1.210	164500	146.9	**23.6**	**22.2**	**21.1**	**19.2**	**17.8**	**16.6**	**15.7**	**14.2**	**13.0**
					28.8	27.2	25.8	23.5	21.7	20.3	19.2	17.3	15.9

Table No. 37 (*Continued*).—STRENGTH OF FLOOR-SLABS.

Bold-faced type, $M=\frac{1}{8}wl^2$; light-faced type, $M=\frac{1}{12}wl^2$.

11. $f_c=700$ $f_s=16,000$ $R=120$ $p=.0087$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	Span in Feet for Given Net Loads per Square Foot of Floor in Pounds. 50	75	100	150	200	250	300	400	500
2	¾	.130	2300	24.3	**4.5**	**3.9**	**3.5**	**2.9**	**2.6**	**2.3**	**2.1**	**1.9**	**1.7**
					5.5	4.8	4.3	3.6	3.2	2.8	2.6	2.3	2.1
2½	¾	.182	4400	30.5	**6.1**	**5.3**	**4.8**	**4.0**	**3.6**	**3.2**	**3.0**	**2.6**	**2.4**
					7.5	6.5	5.9	4.9	4.4	3.9	3.7	3.2	2.9
3	¾	.234	7300	36.6	**7.5**	**6.6**	**6.0**	**5.1**	**4.5**	**4.1**	**3.8**	**3.3**	**3.0**
					9.2	8.1	7.4	6.2	5.5	5.0	4.6	4.0	3.7
3½	¾	.286	10900	42.9	**8.9**	**7.9**	**7.2**	**6.2**	**5.5**	**5.0**	**4.6**	**4.0**	**3.7**
					10.9	9.7	8.8	7.6	6.7	6.1	5.6	4.9	4.5
4	1	.312	13000	48.8	**9.3**	**8.4**	**7.6**	**6.6**	**5.9**	**5.4**	**5.0**	**4.4**	**4.0**
					11.3	10.3	9.3	8.1	7.2	6.6	6.1	5.4	4.9
5	1	.416	23100	61.0	**11.8**	**10.6**	**9.8**	**8.5**	**7.7**	**7.0**	**6.5**	**5.8**	**5.2**
					14.4	12.9	12.0	10.4	9.4	8.6	8.0	7.1	6.4
6	1¼	.494	32600	73.2	**13.3**	**12.1**	**11.2**	**9.9**	**8.9**	**8.2**	**7.6**	**6.8**	**6.2**
					16.2	14.8	13.7	12.1	10.9	10.0	9.3	8.3	7.6
7	1¼	.598	47700	85.4	**15.4**	**14.1**	**13.1**	**11.6**	**10.6**	**9.8**	**9.1**	**8.1**	**7.4**
					18.8	17.2	16.0	14.2	12.9	12.0	11.1	9.9	9.0
8	1¼	.702	65800	97.7	**17.3**	**15.9**	**14.9**	**13.3**	**12.1**	**11.2**	**10.5**	**9.4**	**8.6**
					21.1	19.4	18.2	16.2	14.8	13.7	12.8	11.5	10.5
9	1½	.780	81200	109.9	**18.4**	**17.1**	**16.1**	**14.4**	**13.2**	**12.3**	**11.5**	**10.3**	**9.4**
					23.0	20.9	19.7	17.6	16.1	15.0	14.1	12.6	11.5
10	1½	.884	104300	122.1	**20.1**	**18.8**	**17.7**	**16.0**	**14.7**	**13.7**	**12.9**	**11.6**	**10.6**
					24.6	23.0	21.6	19.5	18.0	16.8	15.8	14.2	12.9
12	1½	1.092	159200	146.6	**23.2**	**21.9**	**20.8**	**18.9**	**17.5**	**16.4**	**15.4**	**14.0**	**12.8**
					28.3	26.8	25.5	23.1	21.4	20.0	18.8	17.0	15.7

12. $f_c=700$ $f_s=18,000$ $R=113$ $p=.0072$

Total Thickness of Slab, Inches.	Thickness of Concrete below Steel, Inches.	Required Area of Steel per Foot of Slab, Sq. In.	Moment of Resistance per Foot of Slab, In.-lbs.	Weight of Slab per Square Foot, Lbs.	50	75	100	150	200	250	300	400	500
2	¾	.107	2100	24.3	**4.4**	**3.8**	**3.4**	**2.8**	**2.5**	**2.3**	**2.1**	**1.8**	**1.6**
					5.4	4.6	4.2	3.4	3.1	2.8	2.6	2.2	2.0
2½	¾	.150	4200	30.4	**5.9**	**5.1**	**4.6**	**3.9**	**3.5**	**3.1**	**2.9**	**2.5**	**2.3**
					7.2	6.2	5.6	4.8	4.3	3.8	3.6	3.1	2.8
3	¾	.193	6900	36.4	**7.3**	**6.4**	**5.8**	**5.0**	**4.4**	**4.0**	**3.7**	**3.2**	**2.9**
					8.9	7.8	7.1	6.1	5.4	4.9	4.5	3.9	3.6
3½	¾	.236	10300	42.7	**8.6**	**7.6**	**6.9**	**6.0**	**5.3**	**4.8**	**4.5**	**3.9**	**3.5**
					10.5	9.3	8.4	7.4	6.5	5.9	5.5	4.8	4.3
4	1	.258	12200	48.6	**9.1**	**8.1**	**7.4**	**6.4**	**5.7**	**5.2**	**4.8**	**4.3**	**3.8**
					11.1	9.9	9.0	7.8	7.0	6.4	5.9	5.3	4.6
5	1	.344	21700	60.8	**11.4**	**10.3**	**9.5**	**8.3**	**7.5**	**6.8**	**6.3**	**5.6**	**5.1**
					13.9	12.6	11.6	10.2	9.2	8.3	7.7	6.9	6.2
6	1¼	.408	30600	72.9	**12.8**	**11.7**	**10.9**	**9.6**	**8.6**	**7.9**	**7.4**	**6.6**	**6.0**
					15.7	14.3	13.3	11.7	10.5	9.7	9.0	8.1	7.4
7	1¼	.494	44800	85.4	**14.8**	**13.6**	**12.7**	**11.3**	**10.2**	**9.4**	**8.8**	**7.8**	**7.1**
					18.1	16.6	15.5	13.8	12.5	11.5	10.8	9.6	8.7
8	1¼	.580	61800	97.9	**16.7**	**15.4**	**14.5**	**12.9**	**11.7**	**10.9**	**10.2**	**9.1**	**8.3**
					20.4	18.8	17.7	15.8	14.3	13.3	12.5	11.1	10.2
9	1½	.644	76300	109.6	**17.9**	**16.6**	**15.6**	**14.0**	**12.8**	**11.9**	**11.1**	**10.0**	**9.1**
					21.9	20.3	19.1	17.1	15.7	14.5	13.6	12.2	11.1
10	1½	.730	98000	121.8	**19.5**	**18.2**	**17.2**	**15.5**	**14.3**	**13.3**	**12.5**	**11.2**	**10.3**
					23.8	22.2	21.0	19.0	17.5	16.2	15.3	13.7	12.6
12	1½	.902	149500	146.3	**22.6**	**21.2**	**20.1**	**18.4**	**17.0**	**15.9**	**15.0**	**13.5**	**12.4**
					27.6	25.9	24.5	22.5	20.8	19.4	18.3	16.5	15.2

APPENDIX A.

ABSTRACT FROM REPORT OF JOINT COMMITTEE ON CONCRETE AND REINFORCED CONCRETE.

CHAPTERS VII AND VIII ON DESIGN AND WORKING STRESSES.*

CHAPTER VII.

DESIGN.

1. **Massive Concrete.**—In the design of massive or plain concrete, no account should be taken of the tensile strength of the material, and sections should usually be proportioned so as to avoid tensile stresses except in slight amounts to resist indirect stresses. This will generally be accomplished in the case of rectangular shapes if the line of pressure is kept within the middle third of the section, but in very large structures, such as high masonry dams, a more exact analysis may be required. Structures of massive concrete are able to resist unbalanced lateral forces by reason of their weight; hence the element of weight rather then strength often determines the design. A leaner and relatively cheap concrete, therefore, will often be suitable for massive concrete structures.

It is desirable generally to provide joints at intervals to localize the effect of contraction (Chap. VI, Sect. 1).

Massive concrete is suitable for dams, retaining walls, and piers in which the ratio of length to least width is relatively small. Under ordinary conditions this ratio should not exceed four. It is also suitable for arches of moderate span.

2. **Reinforced Concrete.**—The use of metal reinforcement is particularly advantageous in members such as beams in which both tension and compression exist, and in columns where the principal stresses are compressive and where there also may be cross-bending. Therefore the theory of design here presented relates mainly to the analysis of beams and columns.

* The cross references refer to Chapter and Section of Joint Committee Report.

3. General Assumptions.—(*a*) *Loads.*—The forces to be resisted are those due to:

1. *The dead load,* which includes the weight of the structure and fixed loads and forces.
2. *The ive load,* or the loads and forces which are variable. The dynamic effect of the live load will often require consideration. Allowance for the latter is preferably made by a proportionate increase in either the live load or the live-load stresses. The working stresses hereinafter recommended are intended to apply to the equivalent static stresses thus determined.

   In the case of high buildings the live load on columns may be reduced in accordance with the usual practice.

(*b*) *Lengths of Beams and Columns.*—The span length for beams and slabs simply upported should be taken as the distance from center to center of supports, but need not be taken to exceed the clear span plus the depth of beam or slab. For continuous or restrained beams built monolithically into supports the span length may be taken as the clear distance between faces of supports. Brackets should not be considered as reducing the clear span in the sense here intended, except that when brackets which make an angle of 45° or more with the axis of a restrained beam are built monolithically with the beam, the span may be measured from the section where the combined depth of beam and bracket is at least one-third more than the depth of the beam. Maximum negative moments are to be considered as existing at the end of the span as here defined.

When the depth of a restrained beam is greater at its ends than at midspan and the slope of the bottom of the beam at its ends makes an angle of not more than 15° with the direction of the axis of the beam at midspan, the span length may be measured from face to face of supports.

The length of columns should be taken as the maximum unstayed length.

(*c*) *Stresses.*—The following assumptions are recommended as a basis for calculations:

1. Calculations will be made with reference to working stresses and safe loads rather than with reference to ultimate strength and ultimate loads.
2. A plane section before bending remains plane after bending.
3. The modulus of elasticity of concrete in compression is constant within the usual limits of working stresses. The distribution of compressive stress in beams is therefore rectilinear.
4. In calculating the moment of resistance of beams the tensile stresses in the concrete are neglected.

5. The adhesion between the concrete and the reinforcement is perfect. Under compressive stress the two materials are therefore stressed in proportion to their moduli of elasticity.
6. The ratio of the modulus of elasticity of steel to the modulus of elasticity of concrete is taken at 15 except as modified in Chapter VIII, Section 8.
7. Initial stress in the reinforcement due to contraction or expansion of the concrete is neglected.

It is recognized that some of the assumptions given herein are not entirely borne out by experimental data. They are given in the interest of simplicity and uniformity, and variations from exact conditions are taken into account in the selection of formulas and working stresses.

The deflection of a beam depends upon the strength and stiffness developed throughout its length. For calculating deflection a value of 8 for the ratio of the moduli will give results corresponding approximately with the actual conditions.

4. **T-beams.**—In beam and slab construction an effective bond should be provided at the junction of the beam and slab. When the principal slab reinforcement is parallel to the beam, transverse reinforcement should be used extending over the beam and well into the slab.

The slab may be considered an integral part of the beam, when adequate bond and shearing resistance between slab and web of beam is provided, but its effective width shall be determined by the following rules:

(*a*) It shall not exceed one-fourth of the span length of the beam;

(*b*) Its overhanging width on either side of the web shall not exceed six times the thickness of the slab.

In the design of continuous T-beams, due consideration should be given to the compressive stress at the support.

Beams in which the T-form is used only for the purpose of providing additional compression area of concrete should preferably have a width of flange not more than three times the width of the stem and a thickness of flange not less than one-third of the depth of the beam. Both in this form and in the beam and slab form the web stresses and the limitations in placing and spacing the longitudinal reinforcement will probably be controlling factors in design.

5. **Floor Slabs Supported along Four Sides.**—Floor slabs having the supports extending along the four sides should be designed and reinforced as continuous over the supports. If the length of the slab exceeds 1.5 times its width the entire load should be carried by transverse reinforcement.

For uniformly distributed loads on square slabs, one-half the live and

dead load may be used in the calculations of moment to be resisted in each direction. For oblong slabs, the length of which is not greater than one and one-half times their width, the moment to be resisted by the transverse reinforcement may be found by using a proportion of the live and dead load equal to that given by the formula $r=l/b-0.5$, where $l=$ length and $b=$breadth of slab. The longitudinal reinforcement should then be proportioned to carry the remainder of the load.

In placing reinforcement in such slabs account may well be taken of the fact that the bending moment is greater near the center of the slab than near the edges. For this purpose two-thirds of the previously calculated moments may be assumed as carried by the center half of the slab and one-third by the outside quarters.

Loads carried to beams by slabs which are reinforced in two directions will not be uniformly distributed to the supporting beams and the distribution will depend on the relative stiffness of the slab and the supporting beams. The distribution which may be expected ordinarily is a variation of the load in the beam in accordance with the ordinates of a parabola, having its vertex at the middle of the span. For any given design, the probable distribution should be ascertained and the moments in the beam calculated accordingly.

6. **Continuous Beams and Slabs.**—When the beam or slab is continuous over its supports, reinforcement should be fully provided at points of negative moment, and the stresses in concrete recommended in Chapter VIII, Section 4, should not be exceeded. In computing the positive and negative moments in beams and slabs continuous over several supports, due to uniformly distributed loads, the following rules are recommended:

(*a*) For floor slabs the bending moments at center and at support should be taken at $wl^2/12$ for both dead and live loads, where w represents the load per linear unit and l the span length.

(*b*) For beams the bending moment at center and at support for interior spans should be taken at $wl^2/12$, and for end spans it should be taken at $wl^2/10$ for center and interior support, for both dead and live loads.

(*c*) In the case of beams and slabs continuous for two spans only, with their ends restrained, the bending moment both at the central support and near the middle of the span should be taken at $wl^2/10$.

(*d*) At the ends of continuous beams the amount of negative moment which will be developed in the beam will depend on the condition of restraint or fixedness, and this will depend on the form of construction used. In the ordinary cases a moment of $wl^2/16$ may be taken; for small beams running into heavy columns this should be increased, but not to exceed $wl^2/12$.

For spans of unusual length, or for spans of materially unequal length, more exact calculations should be made. Special consideration is also required in the case of concentrated loads.

Even if the center of the span is designed for a greater bending moment than is called for by (*a*) or (*b*), the negative moment at the support should not be taken as less than the values there given.

Where beams are reinforced on the compression side, the steel may be assumed to carry its proportion of stress in accordance with the ratio of moduli of elasticity, Chapter VIII, Section 8. Reinforcing bars for compression in beams should be straight and should be two diameters in the clear from the surface of the concrete. For the positive bending moment, such reinforcement should not exceed 1 per cent of the area of the concrete. In the case of cantilever and continuous beams, tensile and compressive reinforcement over supports should extend sufficiently beyond the support and beyond the point of inflection to develop the requisite bond strength.

In construction made continuous over supports it is important that ample foundations should be provided; for unequal settlements are liable to produce unsightly if not dangerous cracks. This effect is more likely to occur in low structures.

Girders, such as wall girders, which have beams framed into one side only, should be designed to resist torsional moment arising from the negative moment at the end of the beam.

7. Bond Strength and Spacing of Reinforcement. — Adequate bond strength should be provided The formula hereinafter given for bond stresses in beams is for straight longitudinal bars. In beams in which a portion of the reinforcement is bent up near the end, the bond stress at places, in both the straight bars and the bent bars, will be considerably greater than for all the bars straight, and the stress at some point may be several times as much as that found by considering the stress to be uniformly distributed along the bar. In restrained and cantilever beams full tensile stress exists in the reinforcing bars at the point of support and the bars should be anchored in the support sufficiently to develop this stress.

In case of anchorage of bars, an additional length of bar should be provided beyond that found on the assumption of uniform bond stress, for the reason that before the bond resistance at the end of the bar can be developed the bar may have begun to slip at another point and "running" resistance is less than the resistance before slip begins.

Where high bond resistance is required, the deformed bar is a suitable means of supplying the necessary strength. But it should be recognized that even with a deformed bar initial slip occurs at early loads,

and that the ultimate loads obtained in the usual tests for bond resistance may be misleading. Adequate bond strength throughout the length of a bar is preferable to end anchorage, but, as an additional safeguard, such anchorage may properly be used in special cases. Anchorage furnished by short bends at a right angle is less effective than by hooks consisting of turns through 180°.

The lateral spacing of parallel bars should be not less than three diameters from center to center, nor should the distance from the side of the beam to the center of the nearest bar be less than two diameters. The clear spacing between two layers of bars should be not less than 1 inch. The use of more than two layers is not recommended, unless the layers are tied together by adequate metal connections, particularly at and near points where bars are bent up or bent down. Where more than one layer is used at least all bars above the lower layer should be bent up and anchored beyond the edge of the support.

8. **Diagonal Tension and Shear.** — When a reinforced concrete beam is subjected to flexural action, diagonal tensile stresses are set up. A beam without web reinforcement will fail if these stresses exceed the tensile strength of the concrete. When web reinforcement, made up of stirrups or of diagonal bars secured to the longitudinal reinforcement, or of longitudinal reinforcing bars bent up at several points, is used, new conditions prevail, but even in this case at the beginning of loading the diagonal tension developed is taken principally by the concrete, the deformations which are developed in the concrete permitting but little stress to be taken by the web reinforcement. When the resistance of the concrete to the diagonal tension is overcome at any point in the depth of the beam, greater stress is at once set up in the web reinforcement.

For homogeneous beams the analytical treatment of diagonal tension is not very complex—the diagonal tensile stress is a function of the horizontal and vertical shearing stresses and of the horizontal tensile stress at the point considered, and as the intensity of these three stresses varies from the neutral axis to the remotest fiber, the intensity of the diagonal tension will be different at different points in the section, and will change with different proportionate dimensions of length to a depth of beam. For the composite structure of reinforced concrete beams, an analysis of the web stresses, and particularly of the diagonal tensile stresses, is very complex; and when the variations due to a change from no horizontal tensile stress in the concrete at remotest fiber to the presence of horizontal tensile stress at some point below the neutral axis are considered, the problem becomes more complex and indefinite. Under these circumstances, in designing recourse is had to the use of the calculated vertical shearing

stress as a means of comparing or measuring the diagonal tensile stresses developed, it being understood that the vertical shearing stress is not the numerical equivalent of the diagonal tensile stress, and that there is not even a constant ratio between them. It is here recommended that the maximum vertical shearing stress in a section be used as a means of comparison of the resistance to diagonal tensile stress developed in the concrete in beams not having web reinforcement.

Even after the concrete has reached its limit of resistance to diagonal tension, if the beam has web reinforcement, conditions of beam action will continue to prevail, at least through the compression area, and the web reinforcement will be called on to resist only a part of the web stresses. From experiments with beams it is concluded that it is safe practice to use only two-thirds of the external vertical shear in making calculations of the stresses that come on stirrups, diagonal web pieces, and bent-up bars, and it is here recommended for calculations in designing that two-thirds of the external vertical shear be taken as producing stresses in web reinforcement.

It is well established that vertical members attached to or looped about horizontal members, inclined members secured to horizontal members in such a way as to insure against slip, and the bending of a part of the longitudinal reinforcement at an angle, will increase the strength of a beam against failure by diagonal tension, and that a well-designed and well-distributed web reinforcement may under the best conditions increase the total vertical shear carried to a value as much as three times that obtained when the bars are all horizontal and no web reinforcement is used.

When web reinforcement comes into action as the principal tension web resistance, the bond stresses between the longitudinal bars and the concrete are not distributed as uniformly along the bars as they otherwise would be, but tend to be concentrated at and near stirrups, and at and near the points where bars are bent up. When stirrups are not rigidly attached to the longitudinal bars, and the proportioning of bars and stirrup spacing is such that local slip of bars occur at stirrups, the effectiveness of the stirrups is impaired, though the presence of stirrups still gives an element of toughness against diagonal tension failure.

Sufficient bond resistance between the concrete and the stirrups or diagonals must be provided in the compression area of the beam.

The longitudinal spacing of vertical stirrups should not exceed one-half the depth of beam, and that of inclined members should not exceed three-fourths of the depth of beam.

Bending of longitudinal reinforcing bars at an angle across the web of the beam may be considered as adding to diagonal tension resistance for a horizontal distance from the point of bending equal to three-fourths

of the depth of beam. Where the bending is made at two or more points the distance between points of bending should not exceed three-fourths of the depth of the beam. In the case of a restrained beam the effect of bending up a bar at the bottom of the beam in resisting diagonal tension may not be taken as extending beyond a section at the point of inflection, and the effect of bending down a bar in the region of negative moment may be taken as extending from the point of bending down of bar nearest the support to a section not more than three-fourths of the depth of beam beyond the point of bending down of bar farthest from the support but not beyond the point of inflection. In case stirrups are used in the beam away from the region in which the bent bars are considered effective, a stirrup should be placed not farther than a distance equal to one-fourth the depth of beam from the limiting sections defined above. In case the web resistance required through the region of bent bars is greater than that furnished by the bent bars, sufficient additional web reinforcement in the form of stirrups or attached diagonals should be provided. The higher resistance to diagonal tension stresses given by unit frames having the stirrups and bent-up bars securely connected together both longitudinally and laterally is worthy of recognition. It is necessary that a limit be placed on the amount of shear which may be allowed in a beam; for when web reinforcement sufficiently efficient to give very high web resistance is used, at the higher stresses the concrete in the beam becomes checked and cracked in such a way as to endanger its durability as well as its strength.

The section to be taken as the critical section in the calculation of shearing stresses will generally be the one having the maximum vertical shear, though experiments show that the section at which diagonal tension failures occur is not just at a support even though the shear at the latter point be much greater.

In the case of restrained beams, the first stirrup or the point of bending down of bar should be placed not farther then one-half of the depth of beam away from the face of the support.

It is important that adequate bond strength or anchorage be provided to develop fully the assumed strength of all web reinforcement.

Low bond stresses in the longitudinal bars are helpful in giving resistance against diagonal tension failures and anchorage of longitudinal bars at the ends of the beams or in the supports is advantageous.

It should be noted that it is on the tension side of a beam that diagonal tension develops in a critical way, and that proper connection should always be made between stirrups or other web reinforcement and the longitudinal tension reinforcement, whether the latter is on the lower side of the beam or on its upper side. Where negative moment exists, as is the

case near the supports in a continuous beam, web reinforcement to be effective must be looped over or wrapped around or be connected with the longitudinal tension reinforcing bars at the top of the beam in the same way as is necessary at the bottom of the beam at sections where the bending moment is positive.

Inasmuch as the smaller the longitudinal deformations in the horizontal reinforcement are, the less the tendency for the formation of diagonal cracks, the beam will be strengthened against diagonal tension failure by so arranging and proportioning the horizontal reinforcement that the unit stresses at points of large shear shall be relatively low.

It does not seem feasible to make a complete analysis of the action of web reinforcement, and more or less empirical methods of calculation are therefore employed. Limiting values of working stresses for different types of web reinforcement are given in Chapter VIII, Section 5. The conditions apply to cases commonly met in design. It is assumed that adequate bond resistance or anchorage of all web reinforcement will be provided.

When a flat slab rests on a column, or a column bears on a footing, the vertical shearing stresses in the slab or footing immediately adjacent to the column are termed punching shearing stresses. The element of diagonal tension, being a function of the bending moment as well as of shear, may be small in such cases, or may be otherwise provided for. For this reason the permissible limit of stress for punching shear may be higher than the allowable limit when the shearing stress is used as a means of comparing diagonal tensile stress. The working values recommended are given in Chapter VIII, Section 5.

9. **Columns.**—By columns are meant compression members of which the ratio of unsupported length to least width exceeds about four, and which are provided with reinforcement of one of the forms hereafter described.

It is recommended that the ratio of unsupported length of column to its least width be limited to 15.

The effective area of hooped columns or columns reinforced with structural shapes shall be taken as the area within the circle enclosing the spiral or the polygon enclosing the structural shapes.

Columns may be reinforced by longitudinal bars; by bands, hoops, or spirals, together with longitudinal bars; or by structural forms which are sufficiently rigid to have value in themselves as columns. The general effect of closely spaced hooping is to greatly increase the toughness of the column and to add to its ultimate strength, but hooping has little effect on its behavior within the limit of elasticity. It thus renders the concrete a safer and more reliable material, and should permit the use of

a somewhat higher working stress. The beneficial effects of toughening are adequately provided by a moderate amount of hooping, a larger amount serving mainly to increase the ultimate strength and the deformation possible before ultimate failure.

Composite columns of structural steel and concrete in which the steel forms a column by itself should be designed with caution. To classify this type as a concrete column reinforced with structural steel is hardly permissible, as the steel generally will take the greater part of the load. When this type of column is used, the concrete should not be relied upon to tie the steel units together nor to transmit stresses from one unit to another. The units should be adequately tied together by tle plates or lattice bars, which, together with other details, such as splices, etc., should be designed in conformity with standard practice for structural steel. The concrete may exert a beneficial effect in restraining the steel from lateral deflection and also in increasing the carrying capacity of the column. The proportion of load to be carried by the concrete will depend on the form of the column and the method of construction. Generally, for high percentages of steel, the concrete will develop relatively low unit stresses, and caution should be used in placing dependence on the concrete.

The following recommendations are made for the relative working stresses in the concrete for the several types of columns:

(*a*) Columns with longitudinal reinforcement to the extent of not less than 1 per cent and not more than 4 per cent, and with lateral ties of not less than $\frac{1}{4}$ inch in diameter 12 inches apart nor more than 16 diameters of the longitudinal bar: the unit stress recommended for axial compression, on concrete piers having a length not more than four diameters, in Chapter VIII, Section 3.

(*b*) Columns reinforced with not less than 1 per cent and not more than 4 per cent of longitudinal bars and with circular hoops or spirals not less than 1 per cent of the volume of the concrete and as hereinafter specified: a unit stress 55 per cent higher than given for (*a*), provided the ratio of unsupported length of column to diameter of the hooped core is not more than 10.

The foregoing recommendations are based on the following conditions:

It is recommended that the minimum size of columns to which the working stresses may be applied be 12 inches out to out.

In all cases longitudinal reinforcement is assumed to carry its proportion of stress in accordance with Section 3 (*c*) 6 of this chapter. The hoops or bands are not to be counted on directly as adding to the strength of the column.

Longitudinal reinforcement bars should be maintained straight, and should have sufficient lateral support to be securely held in place until the concrete has set.

Where hooping is used, the total amount of such reinforcement shall be not less than 1 per cent of the volume of the column, enclosed. The clear spacing of such hooping shall be not greater than one-sixth the diameter of the enclosed column and preferably not greater than one-tenth, and in no case more than $2\frac{1}{2}$ inches. Hooping is to be circular and the ends of bands must be united in such a way as to develop their full strength. Adequate means must be provided to hold bands or hoops in place so as to form a column, the core of which shall be straight and well centered. The strength of hooped columns depends very much upon the ratio of length to diameter of hooped core, and the strength due to hooping decreases rapidly as this ratio increases beyond five. The working stresses recommended are for hooped columns with a length of not more than ten diameters of the hooped core.

The Committee has no recommendation to make for a formula for working stresses for columns longer than ten diameters.

Bending stresses due to eccentric loads, such as unequal spans of beams, and to lateral forces, must be provided for by increasing the section until the maximum stress does not exceed the values above specified. Where tension is possible in the longitudinal bars of the column, adequate connection between the ends of the bars must be provided to take this tension.

10. **Reinforcing for Shrinkage and Temperature Stresses.**— When areas of concrete too large to expand and contract freely as a whole are exposed to atmospheric conditions, the changes of form due to shrinkage and to action of temperature are such that cracks may occur in the mass unless precautions are taken to distribute the stresses so as to prevent the cracks altogether or to render them very small. The distance apart of the cracks, and consequently their size, will be directly proportional to the diameter of the reinforcement and to the tensile strength of the concrete, and inversely proportional to the percentage of reinforcement and also to its bond resistance per unit of surface area. To be most effective, therefore, reinforcement (in amount generally not less than $\frac{1}{3}$ of 1 per cent of the gross area) of a form which will develop a high bond resistance should be placed near the exposed surface and be well distributed. Where openings occur the area of cross-section of the reinforcement should not be reduced. The allowable size and spacing of cracks depends on various considerations, such as the necessity for water-tightness, the importance of appearance of the surface, and the atmospheric changes.

The tendency of concrete to shrink makes it necessary, except where expansion is provided for, to thoroughly connect the component parts of the frame of articulated structures, such as floor and wall members in buildings, by the use of suitable reinforcing material. The amount of reinforcement for such connection should bear some relation to the size of the members connected, larger and heavier members, requiring stronger connections. The reinforcing bars should be extended beyond the critical section far enough, or should be sufficiently anchored to develop their full tensile strength.

11. **Flat Slab.**—The continuous flat slab reinforced in two or more directions and built monolithically with the supporting columns (without beams or girders) is a type of construction which is now extensively used and which has recognized advantages for certain types of structures as, for example, warehouses in which large, open floor space is desired. In its construction, there is excellent opportunity for inspecting the position of the reinforcement. The conditions attending depositing and placing of concrete are favorable to securing uniformity and soundness in the concrete. The recommendations in the following paragraphs relate to flat slabs extending over several rows of panels in each direction. Necessarily the treatment is more or less empirical.

The coefficients and moments given relate to uniformly distributed loads.

(*a*) *Column Capital.*—It is usual in flat slab construction to enlarge the supporting columns at their top, thus forming column capitals. The size and shape of the column capital affect the strength of the structure in several ways. The moment of the external forces which the slab is called upon to resist is dependent upon the size of the capital; the section of the slab immediately above the upper periphery of the capital carries the highest amount of punching shear; and the bending moment developed in the column by an eccentric or unbalanced loading of the slab is greatest at the under surface of the slab. Generally the horizontal section of the column capital should be round or square with rounded corners. In oblong panels the section may be oval or oblong, with dimensions proportional to the panel dimensions. For computation purposes, the diameter of the column capital will be considered to be measured where its vertical thickness is at least $1\frac{1}{2}$ inches, provided the slope of the capital below this point nowhere makes an angle with the vertical of more than 45°. In case a cap is placed above the column capital, the part of this cap within a cone made by extending the lines of the column capital upward at the slope of 45° to the bottom of the slab or dropped panel may be considered as part of the column capital in determining the diameter for design purposes. Without attempting to limit the size of the column capital for

special cases, it is recommended that the diameter of the column capital (or its dimension parallel to the edge of the panel) generally be made not less than one-fifth of the dimension of the panel from center to center of adjacent columns. A diameter equal to 0.225 of the panel length has been used quite widely and acceptably. For heavy loads or large panels especial attention should be given to designing and reinforcing the column capital with respect to compressive stresses and bending moments. In the case of heavy loads or large panels, and where the conditions of the panel loading or variations in panel length or other conditions cause high bending stresses in the column, and also for column capitals smaller than the size herein recommended, especial attention should be given to designing and reinforcing the column capital with respect to compression and to rigidity of connection to floor slab.

(*b*) *Dropped Panel.*—In one type of construction the slab is thickened throughout an area surrounding the column capital. The square or oblong of thickened slab thus formed is called a dropped panel or a drop. The thickness and the width of the dropped panel may be governed by the amount of resisting moment to be provided (the compressive stress in the concrete being dependent upon both thickness and width), or its thickness may be governed by the resistance to shear required at the edge of the column capital and its width by the allowable compressive stresses and shearing stresses in the thinner portion of the slab adjacent to the dropped panel. Generally, however, it is recommended that the width of the dropped panel be at least four-tenths of the corresponding side of the panel as measured from center to center of columns, and that the offset in thickness be not more than five-tenths of the thickness of the slab outside the dropped panel.

(*c*) *Slab Thickness.*—In the design of a slab, the resistance to bending and to shearing forces will largely govern the thickness, and, in the case of large panels with light loads, resistance to deflection may be a controlling factor. The following formulas for minimum thicknesses are recommended as general rules of design when the diameter of the column capital is not less than one-fifth of the dimension of the panel from center to center of adjacent columns, the larger dimension being used in the case of oblong panels. For notation, let

t = total thickness of slab in inches;
L = panel length in feet;
w = sum of live load and dead load in pounds per square foot.

Then, for a slab without dropped panels,
minimum $t=0.024L\sqrt{w}+1\frac{1}{2}$; for a slab with dropped panels, minimum $t=0.02L\sqrt{w}+1$; for a dropped panel whose width is four-tenths of the panel length, minimum $t=0.03L\sqrt{w}+1\frac{1}{2}$.

In no case should the slab thickness be made less than 6 inches, nor should the thickness of a floor be made less than one-thirty-second of the panel length, nor the thickness of a roof slab less than one-fortieth of the panel length.

(*d*) *Bending and Resisting Moments in Slabs.*—If a vertical section of a slab be taken across a panel along a line midway between columns, and if another section be taken along an edge of the panel parallel to the first section, but skirting the part of the periphery of the column capitals at the two corners of the panels, the moment of the copule formed by the

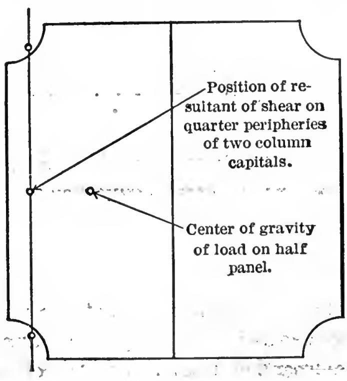


Fig. 1.

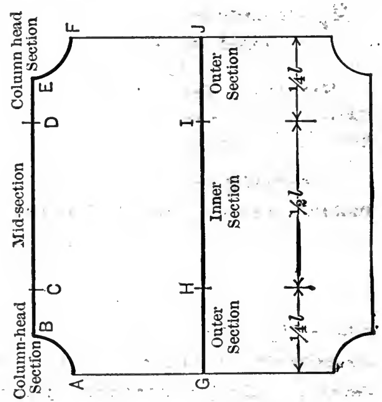


Fig. 2.

external load on the half panel, exclusive of that over the column capital (sum of dead and live load) and the resultant of the external shear or reaction at the support at the two column capitals (see Fig. 1), may be found by ordinary static analysis. It will be noted that the edges of the area here considered are along lines of zero shear except around the column capitals. This moment of the external forces acting on the half panel will be resisted by the numerical sum of (*a*) the moment of the internal stresses at the section of the panel midway between columns (positive resisting moment) and (*b*) the moment of the internal stresses at the section referred to at the end of the panel (negative resisting moment). In the curved portion of the end section (that skirting the column), the stresses considered are the components which act parallel to the normal stresses on the straight portion of the section. Analysis shows that, for a uniformly distributed load, and round columns, and square panels, the numerical sum of the positive moment and the negative moment at the two sections named is given quite closely by the equation

$$M_x = \tfrac{1}{8}wl(l-\tfrac{2}{3}c)^2.$$

In this formula and in those which follow relating to oblong panels,

w = sum of the live and dead load per unit of area;

l = side of a square panel measured from center to center of columns;

l_1 = one side of the oblong panel measured from center to center of columns;

l_2 = other side of oblong panel measured in the same way;

c = diameter of the column capital;

M_x = numerical sum of positive moment and negative moment in one direction;

M_y = numerical sum of positive moment and negative moment in the other direction.

(See paper and closure, Statistical Limitations upon the Steel Requirement in Reinforced Concrete Flat Slab Floors, by John R. Nichols, Jun. Am. Soc. C. E., Transactions Am. Soc. C. E., Vol. LXXVII.)

For oblong panels, the equations for the numerical sums of the positive moment and the negative moment at the two sections named become,

$$M_x = \tfrac{1}{8}wl_2(l_1 - \tfrac{2}{3}c)^2$$

$$M_y = \tfrac{1}{8}wl_1(l_2 - \tfrac{2}{3}c)^2$$

where M_x is the numerical sum of the positive moment and the negative moment for the sections parallel to the dimension l_2, and M_y is the numerical sum of the positive moment and the negative moment for the sections parallel to the dimension l_1.

What proportion of the total resistance exists as positive moment and what as negative moment is not readily determined. The amount of the positive moment and that of the negative moment may be expected to vary somewhat with the design of the slab. It seems proper, however, to make the division of total resisting moment in the ratio of three-eighths for the positive moment to five-eighths for the negative moment.

With reference to variations in stress along the sections, it is evident from conditions of flexure that the resisting moment is not distributed uniformly along either the section of positive moment or that or negative moment. As the law of the distribution is not known definitely, it will be necessary to make an empirical apportionment along the sections; and it will be considered sufficiently accurate generally to divide the sections into two parts and to use an average value over each part of the panel section.

The relatively large breadth of structure in a flat slab makes the effect of local variations in the concrete less than would be the case for narrow members like beams. The tensile resistance of the concrete is less affected by cracks. Measurements of deformations in buildings

under heavy load indicate the presence of considerable tensile resistance in the concrete, and the presence of this tensile resistance acts to decrease the intensity of the compressive stresses. It is believed that the use of moment coefficient somewhat less than those given in a preceding paragraph as derived by analysis is warranted, the calculations of resisting moment and stresses in concrete and reinforcement being made according to the assumptions specified in this report and no change being made in the values of the working stresses ordinarily used. Accordingly, the values of the moments which are recommended for use are somewhat less than those derived by analysis. The values given may be used when the column capitals are round, oval, square, or oblong.

(*e*) *Names for Moment Sections.*—For convenience, that portion of the section across a panel along a line midway between columns which lies within the middle two quarters of the width of the panel (*HI*, Fig. 2) will be called the inner section, and that portion in the two outer quarters of the width of the panel (*GH* and *IJ*, Fig. 2) will be called the outer sections. Of the section which follows a panel edge from column capital to column capital and which includes the quarter peripheries of the edges of two column capitals, that portion within the middle two quarters of the panel width (*CD*, Fig. 2) will be called the mid-section, and the two remaining portions (*ABC* and *DEF*, Fig. 2), each having a projected width equal to one-fourth of the panel width, will be called the column-head sections.

(*f*) *Positive Moment.*—For a square interior panel, it is recommended that the positive moment for a section in the middle of a panel extending across its width be taken as $1/15wl(l-2/3c)^2$. Of this moment, at least 25 per cent should be provided for in the inner section; in the two outer sections of the panel at least 55 per cent of the specified moment should be provided for in slabs not having dropped panels, and at least 60 per cent in slabs having dropped panels, except that in calculations to determine necessary thickness of slab away from the dropped panel at least 70 per cent of the positive moment should be considered as acting in the two outer sections.

(*g*) *Negative Moment.*—For a square interior panel, it is recommended that the negative moment for a section which follows a panel edge from column capital to column capital and which includes the quarter peripheries of the edges of the two column capitals (the section altogether forming the projected width of the panel) be taken as $1/15wl(l-2/3c)^2$. Of this negative moment, at least 20 per cent should be provided for in the mid-section and at least 65 per cent in the two column-head sections of the panel, except that in slabs having dropped panels at least 80 per cent of the specified negative moment should be provided for in the two column-head sections of the panel.

(*h*) *Moments for Oblong Panels.*—When the length of a panel does not exceed the breadth by more than 5 per cent, computation may be made on the basis of a square panel with sides equal to the mean of the length and the breadth.

When the long side of an interior oblong panel exceeds the short side by more than one-twentieth and by not more than one-third of the short side, it is recommended that the positive moment be taken as $1/25wl_2(l_1-2/3c)^2$ on a section parallel to the dimension l_2, and $1/25wl_1(l_2-2/3c)^2$ on a section parallel to the dimension l_1; and that the negative moment be taken as $1/15wl_2(l_1-2/3c)^2$ on a section at the edge of the panel corresponding to the dimension l_2, and $1/15wl_1(l_2-2/3c)^2$ at a section in the other direction. The limitations of the apportionment of moment between inner section and outer section and between mid-section and column-head sections may be the same as for square panels.

(*i*) *Wall Panels.*—The coefficient of negative moment at the first row of columns away from the wall should be increased 20 per cent over that required for interior panels, and likewise the coefficient of positive moment at the section half way to the wall should be increased by 20 per cent. If girders are not provided along the wall or the slab does not project as a cantilever beyond the column line, the reinforcement parallel to the wall for the negative moment in the column-head section and for the positive moment in the outer section should be increased by 20 per cent. If the wall is carried by the slab this concentrated load should be provided for in the design of the slab. The coefficient of negative moments at the wall to take bending in the direction perpendicular to the wall line may be determined by the conditions of restraint and fixedness as found from the relative stiffness of columns, and slab but in no case should it be taken as less than one-half of that for interior panels.

(*j*) *Reinforcement.*—In the calculation of moments all the reinforcing bars which cross the section under consideration and which fulfil the requirements given under Paragraph (*l*) of this chapter may be used. For a column-head section reinforcing bars parallel to the straight portion of the section do not contribute to the negative resisting moment for the column-head section in question. In the case of four-way reinforcement the sectional area of the diagonal bars multiplied by the sine of the angle between the diagonal of the panel and the straight portion of the section under consideration may be taken to act as reinforcement in a rectangular direction.

(*k*) *Point of Inflection.*—For the purpose of making calculations of moments at sections away from the sections of negative moment and positive moment already specified, the point of inflection on any line parallel to a panel edge may be taken as one-fifth of the clear distance on

that line between the two sections of negative moment at the opposite ends of the panel indicated in Paragraph (*e*), of this chapter. For slabs having dropped panels the coefficient of one-fourth should be used instead of one-fifth.

(*l*) *Arrangement of Reinforcement.*—The design should include adequate provision for securing the reinforcement in place so as to take not only the maximum moments by the moments at intermediate sections. All bars in rectangular bands or diagonal bands should extend on each side of a section of maximum moment, either positive or negative, to points at least twenty diameters beyond the point of inflection as defined herein or be hooked or anchored at the point of inflection. In addition to this provision bars in diagonal bands used as reinforcement for negative moment should extend on each side of a line drawn through the column center at right angles to the direction of the band at least a distance equal to thirty-five one-hundredths of the panel length, and bars in diagonal bands used as reinforcement for positive moment should extend on each side of a diagonal through the center of the panel at least a distance equal to thirty-five one-hundredths of the panel length; and no splice by lapping should be permitted at or near regions of maximum stress except as just described. Continuity of reinforcing bars is considered to have advantages, and it is recommended that not more than one-third of the reinforcing bars in any direction be made of a length less than the distance center to center of columns in that direction. Continuous bars should not all be bent up at the same point of their length, but the zone in which this bending occurs should extend on each side of the assumed point of inflection, and should cover a width of at least one-fifteenth of the panel length. Mere draping of the bars should not be permitted. In four-way reinforcement the position of the bars in both diagonal and rectangular directions may be considered in determining whether the width of zone of bending is sufficient.

(*m*) *Reinforcement at Construction Joints.*—It is recommended that at construction joints extra reinforcing bars equal in section to 20 per cent of the amount necessary to meet the requirements for moments at the section where the joint is made be added to the reinforcement, these bars to extend not less than 50 diameters beyond the joint on each side.

(*n*) *Tensile and Compressive Stresses.*—The usual method of calculating the tensile and compressive stresses in the concrete and in the reinforcement, based on the assumptions for internal stresses given in this chapter, should be followed. In the case of the dropped panel the section of the slab and dropped panel may be considered to act integrally for a width equal to the width of the column-head section.

(*o*) *Provision for Diagonal Tension and Shear.*—In calculations for

the shearing stress which is to be used as the means of measuring the resistance to diagonal tension stress, it is recommended that the total vertical shear on two column-head sections constituting a width equal to one-half the lateral dimension of the panel, for use in the formula for determining critical shearing stresses, be considered to be one-fourth of the total dead and live load on a panel for a slab of uniform thickness, and to be three-tenths of the sum of the dead and live loads on a panel for a slab with dropped panels. The formula for shearing unit stress given in Chapter X of this report may then be written $v=\frac{0.25W}{bjd}$ for slabs of uniform thickness, and $v=\frac{0.30W}{bjd}$ for slabs with dropped panels, where W is the sum of the dead and live load on a panel, b is half the lateral dimension of the panel measured from center to center of columns, and jd is the lever arm of the resisting couple at the section.

The calculation of what is commonly called punching shear may be made on the sssumption of a uniform distribution over the section of the slab around the periphery of the column capital and also of a uniform distribution over the section of the slab around the periphery of the dropped panel, using in each case an amount of vertical shear greater by 25 per cent than the total vertical shear on the section under consideration.

The values of working stresses should be those recommended for diagonal tension and shear in Chapter VIII, Section 5.

(*p*) *Walls and Openings.*—Girders or beams should be constructed to carry walls and other concentrated loads which are in excess of the working capacity of the slab. Beams should also be provided in case openings in the floor reduce the working strength of the slab below the required carrying capacity.

(*q*) *Unusual Panels.*—The coefficients, apportionments, and thicknesses recommended are for slabs which have several rows of panels in each direction, and in which the size of the panels is approximately the same. For structures having a width of one, two, or three panels, and also for slabs having panels of markedly different sizes, an analysis should be made of the moments developed in both slab and columns, and the values given herein modified accordingly. Slabs with paneled ceiling or with depressed paneling in the floor are to be considered as coming under the recommendations herein given.

(*c*) *Bending Moments in Columns.*—Provision should be made in both wall columns and interior columns for the bending moment which will be developed by unequally loaded panels, eccentric loading, or uneven spacing of columns. The amount of moment to be taken by a column will depend upon the relative stiffness of columns and slab, and computa-

tions may be made by rational methods, such as the principle of least work, or of slope and deflection. Generally, the larger part of the unequalized negative moment will be transmitted to the columns, and the column should be designed to resist this bending moment. Especial attention should be given to wall columns and corner columns.

WORKING STRESSES.

1. General Assumptions.—The following working stresses are recommended for static loads. Proper allowances for vibration and impact are to be added to live loads where necessary to produce an equivalent static load before applying the unit stresses in proportioning parts.

In selecting the permissible working stress on concrete, the designer should be guided by the working stresses usually allowed for other materials of construction, so that all structures of the same class composed of different materials may have approximately the same degree of safety.

The following recommendations as to allowable stresses are given in the form of percentages of the ultimate strength of the particular concrete which is to be used; this ultimate strength is that developed at an age of 28 days, in cylinders 8 inches in diameter and 16 inches long, of the consistency described in Chapter IV, Section 2 (*d*), made and stored under laboratory conditions. In the absence of definite knowledge in advance of construction as to just what strength may be expected, the Committee submits the following values as those which should be obtained with materials and workmanship in accordance with the recommendations of this report.

Although occasional tests may show higher results than those here given, the Committee recommends that these values should be the maximum used in design.

TABLE OF COMPRESSIVE STRENGTHS OF DIFFERENT MIXTURES OF CONCRETE.

(In Pounds per Square Inch.)

Aggregate	1 : 3*	1 : 4½*	1 : 6*	1 : 7½*	1 : 9*
Granite, trap rock	3300	2800	2200	1800	1400
Gravel, hard limestone and hard sandstone	3000	2500	2000	1600	1300
Soft limestone and sandsone	2200	1800	1500	1200	1000
Cinders	800	700	600	500	400

* Combined volume fine and coarse aggregate measured separately.

NOTE.—For variations in the moduli of elasticity, see Chapter VIII, Section 8.

2. **Bearing.**—When compression is applied to a surface of concrete of at least twice the loaded area, a stress of 35 per cent of the compressive strength may be allowed in the area actually under load.

3. **Axial Compression.**—For concentric compression on a plain concrete pier, the length of which does not exceed 4 diameters, or on a column reinforced with longitudinal bars only, the length of which does not exceed 12 diameters, 22.5 per cent of the compressive strength may be allowed.

For other forms of columns the stresses obtained from the ratios given in Chapter VII, Section 9, may govern.

4. **Compression in Extreme Fiber.**—The extreme fiber stress of a beam, calculated on the assumption of a constant modulus of elasticity for concrete under working stresses may be allowed to reach 32.5 per cent of the compressive strength. Adjacent to the support of continuous beams stresses 15 per cent higher may be used.

5. **Shear and Diagonal Tension.**—In calculations on beams in which the maximum shearing stress in a section is used as the means of measuring the resistance to diagonal tension stress, the following allowable values for the maximum vertical shearing stress in concrete, calculated by the method given in Chapter X, Formula 22, are recommended:

(*a*) For beams with horizontal bars only and without web reinforcement, 2 per cent of the compressive strength.

(*b*) For beams with web reinforcement consisting of vertical stirrups looped about the longitudinal reinforcing bars in the tension side of the beam and spaced horizontally not more than one-half the depth of the beam; or for beams in which longitudinal bars are bent up at an angle of not more than 45° or less than 20° with the axis of the beam, and the points of bending are spaced horizontally not more than three-quarters of the depth of the beam apart, not to exceed 4½ per cent of the compressive strength.

(*c*) For a combination of bent bars and vertical stirrups looped about the reinforcing bars in the tension side of the beam and spaced horizontally not more than one-half of the depth of the beam, 5 per cent of the compressive strength.

(*d*) For beams with web reinforcement (either vertical or inclined) securely attached to the longitudinal bars in the tension side of the beam in such a way as to prevent slipping of bar past the stirrup, and spaced horizontally not more than one-half of the depth of the beam in case of vertical stirrups and not more than three-fourths of the depth of the beam in the case of inclined members, either with longitudinal bars bent up or not, 6 per cent of the compressive strength.

The web reinforcement in case any is used should be proportioned by using two-thirds of the external vertical shear in Formula 24 or 25

in Chapter X. The effect of longitudinal bars bent up at an angle of from 20 to 45° with the axis of the beam may be taken at sections of the beam in which the bent-up bars contribute to diagonal tension resistance as defined under Chapter VII, Section, 8 as reducing the shearing stresses to be otherwise provided for. The amount of reduction of the shearing stress by means of bent-up bars will depend upon their capacity, but in no case should be taken as greater than 4½ per cent of the compressive strength of the concrete over the effective cross-section of the beam (Formula 22). The limit of tensile stress in the bent-up portion of the bar calculated by Formula 25, using in this formula an amount of total shear corresponding to the reduction in shearing stress assumed for the bent-up bars, may be taken as specified for the working stress of steel, but in the calculations the stress in the bar due to its part as longitudinal reinforcement of the beam should be considered. The stresses in stirrups and inclined members when combined with bent-up bars are to be determined by finding the amount of the total shear which may be allowed by reason of the bent-up bars, and subtracting this shear from the total external vertical shear. Two-thirds of the remainder will be the shear to be carried by the stirrups, using Formulas 24 or 25 in Chapter X.

Where punching shear occurs, provided the diagonal tension requirements are met, a shearing stress of 6 per cent of the compressive strength may be allowed.

6. **Bond.**—The bond stress between concrete and plain reinforcing bars may be assumed at 4 per cent of the compressive strength, or 2 per cent in the case of drawn wire. In the best types of deformed bar the bond stress may be increased, but not to exceed 5 per cent of the compressive strength of the concrete.

7. **Reinforcement.**—The tensile or compressive stress in steel should not exceed 16,000 pounds per square inch.

In structural steel members the working stresses adopted by the American Railway Engineering Association are recommended.

8. **Modulus of Elasticity.**—The value of the modulus of elasticity of concrete has a wide range, depending on the materials used, the age, the range of stresses between which it is considered, as well as other conditions. It is recommended that in computations for the position of the neutral axis, and for the resisting moment of beams and for compression of concrete in columns, it be assumed as:

(*a*) One-fortieth that of steel, when the strength of the concrete is taken as not more than 800 pounds per square inch.

(*b*) One-fifteenth that of steel, when the strength of the concrete is taken as greater than 800 pounds per square inch and less than 2200 pounds per square inch.

(*c*) One-twelfth that of steel, when the strength of the concrete is taken as greater than 2200 pounds per square inch and less than 2900 pounds per square inch, and

(*d*) One-tenth that of steel, when the strength of the concrete is taken as greater than 2900 pounds per square inch.

Although not rigorously accurate, these assumptions will give safe results. For the deflection of beams which are free to move longitudinally at the supports, in using formulas for deflection which do not take into account the tensile strength developed in the concrete, a modulus of one-eighth of that of steel is recommended.

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